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Compositions of Involutive Power Series, and Reversible Series

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Dedicated to Walter Hayman on his eightieth birthday

Abstract. For each natural number n, we characterise the invertible series (under composition) that are the composition of n proper involutions. We work with formal power series in one variable over a field of characteristic zero. We also describe the *reversible* series (those conjugate to their own inverses), and the series that are the composition of n reversible series.

Keywords. Formal Power Series, Involution, Reversible, Composition, Group.

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1. Introduction

A reversible map is, by definition, one that is conjugate to its inverse. Reversible maps arise in a number of interesting situations, including the n-body problem, billiards, real and complex analysis, approximation theory, conformal mapping, linear algebra and analytic geometry. One way to make a reversible map is to compose two involutive maps. The natural setting for such matters is group theory. In view of this, we decided to investigate reversibility and products of involutions in various groups (principally ones of infinite dimension). Considering diffeomorphisms in one real variable, or biholomorphic germs in one complex variable, one is led to the group of formally-invertible formal power series in one indeterminate. The present paper reports the result of an investigation of this case. (See below for some more remarks about the wider context.)

We work over an arbitrary field F of characteristic zero. We find that some aspects of reversibility depend on algebraic features of the field F, and in particular on the presence of primitive fourth roots of unity. Analysts will for the most part want to know about the cases $F = \mathbb{C}$ and $F = \mathbb{R}$, but there is no advantage in restricting just to these two cases.

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The set $F[[x]]$ of formal power series forms an F-algebra under coefficientwise sums and convolution product. It is a local ring, a complete valuation ring with respect to the order valuation

$$f \mapsto \mathrm{ord}(f),$$

where the order $\mathrm{ord}(\sum_{n=0}^{\infty} a_n x^n)$ is the least n with $a_n \neq 0$. The algebra $F[x]$ of polynomials is dense in $F[[x]]$ with respect to the metric associated to the valuation.

The maximal ideal, M, consisting of the series with $a_0 = 0$, has additional structure: its elements may be composed formally, and it forms a composition semigroup. We focus here on the group of invertible elements of this semigroup:

Definition 1. We define $G = G(F)$ to be the group of all the series $\sum_{n=0}^{\infty} a_n x^n \in F[[x]]$, with $a_0 = 0$ and $a_1 \neq 0$, equipped with the operation of formal composition. We denote the composition of two series $h, k \in G$ by $h \circ k$.

For some background on formal series, see [4, Chapter 1].

To avoid confusion, we denote compositional powers $\phi \circ \cdots \circ \phi$ by $\phi^{\circ n}$, and ordinary (convolutional) powers by ϕ^n. However, we shall, as is usual in group theory, refer to the group operation as the "product".

An involution in G is an element of order at most two. In what follows, we determine quite explicitly which invertible formal power series arise as the product of two involutive series, and which arise as the product of three, and which arise as the product of any number. We find that a series is a product of involutions if and only if it is such a product "modulo x^4", and we use this to show that each product of any number of involutions may be obtained as the product of four. Then we give an explicit characterisation of the reversible series, and show that each product of any number of reversible series is the composition of two.

Exact statements of the results will be found in Section 3.

We include some classical material and folklore that is not readily accessible, in the interest of completeness and intelligibility. Some of the results about involutions given in Theorem 4 overlap with those of Kasner [9]. See the remarks in Section 8 on this point.

To set our results in a wider context, we observe that there are many classical results about products of involutions. One learns in a first course on group theory that every permutation is the composition of transpositions, and in a first course on complex variables that every Möbius transformation is the composition of two or four inversions. In fact it is not hard to see that each permutation is the composition of two involutions, and each Möbius map is the composition of two involutive Möbius maps. It is quite common that each element of a group can be obtained as the product of a relatively small number of involutions. This can happen even in very large groups. An example is the group of homeomorphisms in one real variable, in which each element is the product of four involutions. (cf. [6, 13, 8]). The same result holds in the group of diffeomorphisms of $\mathbb{R}$.

The author and Patrick Ahern have recently worked on the corresponding problems in the subgroup $\mathcal{G}$ of convergent series $f \in G(\mathbb{C})$, i.e. the group of invertible holomorphic germs in one variable. The conjugacy classification in $\mathcal{G}$ is more complicated than for formal series, and was worked out by Écalle and Voronin (independently) [1, 12]. Centralisers are also more complicated, and generically countable for elements having multipliers of modulus 1. These were characterised by Baker and Liverpool [1, 2, 10]. It turns out that a germ f is reversible in $\mathcal{G}$ if and only if it is reversed by an element conjugate to a rotation of finite even order. Theorem 5 below implies that a reversible germ f is *formally* conjugate to one of the functions

$$g(z) = \frac{\mu z}{(1+\lambda z^p)^{1/p}},$$

with $\mu \in \{\pm 1\}$, $p \in \mathbb{N}$ and $\lambda \in \mathbb{C}$.

2. Notation and Preliminaries

2.1. Various Groups. We denote the identity of G by $e = x + 0x^2 + 0x^3 + \cdots$ and consider the set

$$I = I(G) = \{\tau \in G : \tau^{\circ 2} = e\}$$

of involutions. The condition $\tau^{\circ 2} = e$ translates into an infinite chain of polynomial conditions on the coefficients of τ. The coefficient a_1 of an invertible series (its so-called *multiplier*) $a = \sum_n a_n x^n \in G$ is a conjugacy invariant (i.e. depends only on the conjugacy class of the series a in G). We denote the multiplier of a by $\mu(a)$. The multiplier of an involution must be ± 1, and one checks readily that the only involution with multiplier 1 is e; the others are called *proper involutions*. We consider the set

$$I^{\circ n} = \{\tau_1 \circ \cdots \circ \tau_n : \tau_i \in I\}$$

of products of n involutions ($n = 1,2,\ldots$), the normal subgroups

$$H = \{\phi \in G : \phi \text{ has multiplier } \pm 1\},$$

$$H^+ = \{\phi \in G : \phi \text{ has multiplier } 1\},$$

$$I^\infty = \langle I \rangle = \bigcup_{n=1}^{\infty} I^{\circ n},$$

(the group generated by all the involutions), and

$$I^{\infty +} = I^\infty \cap H^+,$$

the group of products of an even number of proper involutions.

Note that the set of proper involutions is $I \cap (-H^+)$, where $-H^+$ denotes the coset

$$\{\phi \in H : \phi \text{ has multiplier } -1\}.$$

We also consider the corresponding sets and groups "mod x^{k+1}". To be precise:

Definition 2. Given $f, g \in G$ and $k \in \mathbb{N}$, we say that $f = g \mod x^{k+1}$ if f and g have the same coefficients up to and including that of x^k.

For each $k \in \mathbb{N}$, we let

$$J_k = \{f \in G : f = e \mod x^{k+1}\} = \left\{ x + \sum_{n=k+1}^{\infty} a_n x^n : a_n \in F, \forall n \right\},$$

and observe that J_k is a normal subgroup of G. We denote the quotient G/J_k by G_k, and the images under the quotient map of I, $I^{\circ n}$, H, H^+, I^∞ and $I^{\infty+}$ by I_k, $I_k^{\circ n}$, H_k, H_k^+, I_k^∞, and $I_k^{\infty+}$, respectively. Those that are groups are normal subgroups of the F-Lie group G_k.

2.2. Reversible Elements. We say that an element $\phi \in G$ is *reversible* if there exists $h \in G$ such that

$$h^{\circ -1} \circ \phi \circ h = \phi^{\circ -1},$$

and in this case we also say that h *reverses* ϕ.

We denote the set of all reversible elements by $R(G)$, or just R.

We denote the set of all elements $h \in G$ that reverse a given ϕ by R_ϕ.

Observe that the product $h \circ k$ of two elements of R_ϕ lies in the centraliser C_ϕ of ϕ in G, so $C_\phi \cup R_\phi$ forms a subgroup of G, in which C_ϕ has index 1 or 2. Evidently, it has index 1 in two cases: $C_\phi = R_\phi$ if and only if ϕ is an involution, and $R_\phi = \emptyset$ if and only if ϕ is not reversible.

It is elementary (and true in any group) that

$$I^{\circ 2}(G) \subset R(G),$$

and that the following three conditions are equivalent:
(1) $f \in R(G)$;
(2) $f = u^{\circ -1} \circ v$, for some $u, v \in G$ such that $u^{\circ 2} = v^{\circ 2}$;
(3) $f = g \circ h$ and $f^{\circ -1} = h \circ g$, for some $g, h \in G$.

We denote the set of all products of n reversible series by

$$R^{\circ n} = \{\phi_1 \circ \cdots \circ \phi_n : \phi_i \in R\},$$

and the group of all products of any number by

$$R^\infty = \langle R \rangle = \bigcup_{n=1}^{\infty} R^{\circ n}.$$

We observe that

$$I^{\circ 2n} \subset R^{\circ n} \subset R^\infty \le H.$$

2.3. Background: Classical Results. We denote the compositional product $f_1 \circ \cdots \circ f_n$ by $\Pi_{j=1}^{\circ n} f_j$, and use $\Pi_{j=1}^{\circ\infty} f_j$ for the limit $\lim_{n\uparrow\infty} \Pi_{j=1}^{\circ n} f_j$ (if it exists with respect to the valuation metric).

Observe that a sequence $\{g_n\}$ in $F[[x]]$ is convergent with respect to the valuation metric if and only if for each m the m-th coefficient $a_m(g_n)$ is eventually constant for large enough n (depending on m). Thus all the sets I, G, H, H^+, I^∞, $I^{\infty+}$, and J_k are closed in $F[[x]]$, and hence complete with respect to the induced metric.

Lemma 1. *Each $\phi \in G$ is a convergent product*

$$\phi = (\lambda x) \circ \prod_{j=2}^{\circ\infty} (x + a_j x^j),$$

for some λ and $a_j \in F$, with $\lambda \neq 0$.

Proof. One presumes this has been known for over a century (at least for $F = \mathbb{C}$). In any case, it is straightforward. One proves it by progressively converting the inverse $\phi^{\circ-1}$ to the form $x + \sum_{n=j}^{+\infty} b_n x^n$, for larger and larger j, by using right-composition with the factors, for suitable λ and a_j. ∎

Corollary 2. *A continuous function ν mapping G to some topological space is a conjugacy invariant (i.e. is constant on each conjugacy class) if and only if*

$$\nu(\lambda^{-1}\phi(\lambda x)) = \nu(\phi) = \nu\left((x + ax^j)^{\circ-1} \circ \phi \circ (x + ax^j)\right)$$

whenever $\phi \in G$, $0 \neq \lambda \in F$, $2 \leq j \in \mathbb{N}$, and $a_j \in F$.

We note that the coefficient projection maps from $F[[x]]$ to F are all continuous (with respect to *any* metric on F), as are all polynomials depending on a finite number of variables, and indeed all sequences of such polynomials (regarded as maps to the product $\Pi_{n=1}^{\infty} F$).

Membership in each $I^{\circ n}$ is a conjugacy invariant. For $n = 1$, it is determined by an infinite sequence of polynomial identities in the coefficients. These may be put in the form of recursion relations specifying the odd-index coefficients in terms of the preceeding coefficients. The even-index coefficients are unconstrained. We shall see that for $n = 2$ the condition simplifies considerably, and that for $n = 4$ it simplifies radically.

The conjugacy classes of G are well-understood. For our purposes, it suffices to consider the classes contained in H:

For each $p \geq 1$, let F_p denote the multiplicative group of p-th powers of nonzero elements of F, and for $a \in F^\times$ (the multiplicative group of nonzero elements of F), let $[a]_p$ denote the coset aF_p, an element of the quotient $F^\times/F_p$. Take, for instance, $F = \mathbb{R}$. Then when p is even, $[a]_p$ is essentially the sign of a, whereas when p is odd, $[a]_p$ is independent of a.

Lemma 3. *(1) Each $\phi \in H^+$, apart from e, is conjugate (in G) to one of the polynomials*

$$x + ax^{p+1} + \alpha a^2 x^{2p+1},$$

where $0 \neq a \in F$, $p \geq 1$, and $\alpha \in F$. The integer p, the coset $[a]_p$ and $\alpha \in F$ are conjugacy invariants.
(2) Each $\phi \in -H^+$ is conjugate to $-x$ or one of

$$-x + ax^{p+1} - \alpha a^2 x^{2p+1},$$

where $a \in F$, p is even, and $\alpha \in F$. The p, $[a]_p$ and α are conjugacy invariants.

In the sequel, we denote the conjugacy invariant p associated to $\phi \in H$ by $p(\phi)$.

For instance, when $F = \mathbb{C}$ (or any algebraically-closed field), then $[a]_p$ is independent of a, so p and α are the only conjugacy invariants.

This result may be proved by applying Corollary 2 (and we make some remarks related to this below), but is classical. It is already given (for $F = \mathbb{C}$, in somewhat less sharp form) in Kasner's 1916 paper [9, Theorem VI] but we would be surprised if it were not already known at least fifty years earlier. Chen gives it [5], for the case $F = \mathbb{R}$, and Lubin [11] states the general version, describing it as "well-known to analysts".

3. Main Results

The first main result summarises what we know about products of involutions.

Theorem 4.
(1) Let $\phi \in G$. Then ϕ is the product of two proper involutions if and only if ϕ is conjugate to $x + ax^{2m} + ma^2x^{4m-1}$ for some $m \geq 1$ and $a \in F$.
(2) The product of any odd number of proper involutions is the product of three. A series $\phi \in G$ is the product of three proper involutions if and only if $\phi \mod x^4$ is a proper involution modulo x^4, and this happens if and only if $\phi = -x + ax^2 - a^2x^3 \mod x^4$, for some $a \in F$.
(3) The product of any even number of proper involutions is the product of four. A series $\phi \in G$ is such a product if and only if $\phi \mod x^4$ is the product of two proper involutions modulo x^4, and this happens if and only if $\phi = x + ax^2 + a^2x^3 \mod x^4$, for some $a \in F$. Thus

$$I^\infty = I^{\circ 4} = \{\phi : \phi \mod x^4 \in I_3^\infty\}.$$

(4)

$$I_k^{\infty +} \subset \begin{cases} I_k^{\circ 2} &, \quad 1 \leq k \leq 6, \\ I_k^{\circ 4} &, \quad k \geq 7. \end{cases}$$

$$-I_k^{\infty +} \subset \begin{cases} I_k &, \quad 1 \leq k \leq 4, \\ I_k^{\circ 3} &, \quad k \geq 5. \end{cases}$$

$$I_k^{\infty} = \begin{cases} I_k^{\circ 2} & , \quad 1 \le k \le 4, \\ I_k^{\circ 3} & , \quad 5 \le k \le 6, \\ I_k^{\circ 4} & , \quad k \ge 7. \end{cases}$$

The next result is about reversible series, and products of them. It depends on an algebraic aspect of the coefficient field F, so we introduce another item of notation before stating it.

We let $M(F)$ denote the set of integers p such that F has a solution to $\omega^p = -1$. Note that $p \in M(F)$ if and only if $q \in M(F)$, where q is the highest power of 2 that divides p. In particular, $M(F)$ contains all odd natural numbers.

Theorem 5. *Let $\phi \in H$ have multiplier μ $(= \pm 1)$. Then the following three conditions are equivalent:*
(a) $\phi \in R$;
(b) ϕ is conjugate to

$$\frac{\mu x}{(1 + \lambda x^p)^{1/p}},$$

for some $p \in M(F)$ and $\lambda \in F$.
(c) ϕ is conjugate to

$$\mu x + a x^{p+1} + \mu \frac{(p+1)a^2}{2} x^{2p+1},$$

for some $p \in M(F)$ and $a \in F$.

Condition (c) obviously provides the basis for an explicit algorithm to determine whether or not $\phi \in R$. Condition (b) is interesting because it embeds ϕ in a one-parameter subgroup of G, and $\phi^{\circ -1}$ is obtained explicitly by replacing λ by $-\lambda$.

The following corollary of the proof gives rather more abstract characterisations of reversibility, but part (b) sheds some light on the nature of a reversing series.

Corollary 6. *Let $\phi \in G$. Then the following three conditions are equivalent:*
(a) $\phi \in R$;
(b) ϕ is reversed by an element of some finite even order $2p$. The order may be taken to be a power of 2, and the reversing element conjugate to ωx, for some root of unity $\omega \in F$.
(c) $\phi = g^{\circ -1} \circ h$, where $g^{\circ 2} = h^{\circ 2}$ has finite order.

The next two corollaries concern the relationship between R and $I^{\circ 2}$.

Corollary 7. *Let $f \in H$ have $p(f)$ odd. Then $f \in R$ if and only if $f \in I^{\circ 2}$.*

Corollary 8. *$R = I^{\circ 2}$ if and only if F has no primitive 4-th root of 1 (i.e. F has no square root of -1).*

The final theorem describes the product of an arbitrary number of reversible series.

Theorem 9.
(1) Suppose F has no square root of -1. then
$$I^\infty = R^{\circ 2} = R^\infty \neq H,$$
and $f \in R^\infty$ if and only if
$$f = \mu x + a x^2 + \mu a^2 x^3 \mod x^4,$$
for some $\mu = \pm 1$ and $a \in F$.
(2) Suppose F has a square root of -1. Then
$$I^\infty \neq R^{\circ 2} = R^\infty = H.$$

Corollary 10. *The composition of any number of reversible series is the composition of two.*

4. Proof of Theorem 4

We begin by recording a couple of points about the proof of the classical Lemma 3.

To prove (1), one makes use of conjugations with the generators λx and $(x+ax^j)$, for suitable $\lambda \in F^\times$, $j \geq 2$ and $a \in F$. These are used to fix the coefficient of x^{p+1}, then to eliminate the coefficients of x^{p+2},. . .,x^{2p}, and then the coefficients of x^{2p+2} and higher powers. The inverse of $(x+ax^j)$ is given by an explicit formula:
$$(x+ax^j)^{\circ -1} = x + \sum_{t=1}^{\infty} c_t x^{1+it},$$
where $i = j-1$ and the coefficients $c_t(a)$ are given by the recursion
$$c_t = (-a)^t - \sum_{r=1}^{t-1} \binom{ir}{t-r} a^{t-r} c_r.$$
After some manipulation, one obtains the explicit conjugation formula (for $\phi = x + \sum_{m=p+1}^{\infty} a_m x^m$):
$$(x+ax^j)^{-1} \circ \phi \circ (x+ax^j) = x + \sum_{m=p+1}^{\infty} a_m x^m \left\{ \frac{(1+ax^i)^m}{1+ajx^i} \right\} + \mathrm{O}(x^{2p+j}).$$
From this we see that, for $k < p+j$, the effect on the $p+k$-th coefficient when ϕ is conjugated by $x+ax^j$ is to change it from a_{p+k} to a_{p+k} plus a sum of terms of the form $sa^u a_m$, where $s \in \mathbb{Z}$, $u \in \mathbb{N}$, $a_m \neq 0$, $m \geq p+1$, and $m+ui = p+k$. Indeed, s is the coefficient of y^u in $(1+ay)^m/(1+ajy)$. Apart from facilitating the proof of part (1) of Lemma 3, this observation allows us to deduce the following, which makes it easier to evaluate the invariant α.

Lemma 11. *Let $\phi = x + \sum_{n=2}^{\infty} a_n x^n \in H^+$, have invariants p, $[a_{p+1}]_p$ and α. Then:*
(1) The coefficients $a_2, \ldots, a_p$ are zero, and $a_{p+1} \neq 0$.
(2) The invariant α is of the form $a_{2p+1}/a_{p+1}^2 + A/a_{p+1}^b$, where A is a polynomial in $a_{p+1}, \ldots, a_{2p-2}$, having rational coefficients, and b is a positive integer.
(3) If p is odd, and a_n is zero for all odd $n < 2p+1$, then $A = 0$, so

$$\alpha = \frac{a_{2p+1}}{a_p^2}.$$

Proof. The reason for the second part is that for $p+1 < p+j < 2p+1$, one removes a_{p+j} by conjugating with $x + ax^j$, with

$$a = \frac{a_{p+j}}{(i-p)a_{p+1}},$$

and this may alter a_m for $p+j < m \leq 2p+1$ by adding terms of the kind described above. For $j > p$, the same conjugation removes a_{p+j}, but has no further affect on a_{2p+1}.

The reason for the third part is that in the special case where p is odd and $a_m = 0$ for *odd* m between $p+1$ and $2p$, the conjugations required to remove the remaining a_m (for $m \neq p+1$ or $2p+1$) have no effect on the coefficient of x^{2p+1}: only even-indexed coefficients are affected. Thus, in this case, a_{2p+1} already has its 'final' value. ∎

It is readily seen that if a series $\sum a_n x^n$ is a proper involution, then the first $n \geq 2$ with $a_n \neq 0$ (if any) is even (cf. [7]). So the following well-known fact is immediate from (2) in Lemma 3:

Corollary 12. *Each proper involution in G is conjugate to $-x$.*

Corollary 13. *If $\tau \in G$ is a proper involution, then the only involutions that commute with τ are e and τ.*

Proof. The only odd involutions in G are $\pm x$. The centraliser of $-x$ in G is the group of odd elements of G. Thus the only involutions commuting with $-x$ are $\pm x$, and the result follows on conjugating $-x$ to τ. ∎

This shows that G has a rich supply of noncommuting involutions, and $I^{\circ 2}$ is a substantially larger set than I. To get an idea of scale, consider the case $F = \mathbb{R}$, and look at the Lie groups H_k. The two cosets of H_k^+ are affine subspaces of $\mathbb{R}[x]_k$, and the set I_k is an algebraic subvariety of dimension floor$(k/2)$. The map

$$\begin{cases} I_k \times I_k & \to & H_k \\ (\tau_1, \tau_2) & \mapsto & \tau_1 \circ \tau_2, \end{cases}$$

has image of dimension at most 2floor$(k/2) - 1$, and H_k has dimension $k-1$. The map is obviously constant on sets of the form

$$\{(\tau_1 \circ \tau^{\circ -1}, \tau \circ \tau_2) : \tau \in I_k\},$$

but provided the level sets are not significantly larger than these (which seems plausible if you consider the matter) it is reasonable to expect that the image actually has the above dimension and intersects H_k^+ in a set of codimension at most one, depending on the parity of k.

Lemma 14. *Let $\tau = -x + \sum_{n=2m}^{\infty} a_n x^n$ be a proper involution with $a_{2m} \neq 0$. Then $a_n = 0$ for each odd $n < 4m-1$, and $a_{4m-1} = m a_{2m}^2$.*

Proof. This is seen by comparing coefficients in $\tau^{\circ 2} = x$. ■

Corollary 15. *The conjugacy invariants associated to $-\tau$ are $p = 2m-1$, $[a_{p+1}]_p$ and $\alpha = m$.*

Lemma 16.
(1) $I_3 = \{x\} \cup \{-x + ax^2 - a^2x^3 : a \in F\}$.
(2) $\phi \in G_3$ is the product of two proper involutions if and only if

$$\phi = x + ax^2 + a^2x^3$$

for some $a \in F$.
(3) The product of any odd number of proper involutions in G_3 is a proper involution.
(4) $I_3^\infty = I_3^{\circ 2}$.

Proof. Routine calculation shows that

$$\begin{aligned} & (c_1x + c_2x^2 + c_3x^3) \circ (d_1x + d_2x^2 + d_3x^3) \\ = \;& c_1d_1x + (c_1d_2 + c_2d_1^2)x^2 + (c_1d_3 + 2c_2d_1d_2 + c_3d_1^3)x^3 \mod x^4. \end{aligned}$$

Thus for $\epsilon = \pm 1$, we have

$$\begin{aligned} & (\epsilon_1 x + c_2x^2 + \epsilon_1 c_2 x^3) \circ (\epsilon_2 x + d_2x^2 + \epsilon_2 d_2 x^3) \\ = \;& \epsilon_1\epsilon_2 x + (c_2 + \epsilon_1 d_2)x^2 + \epsilon_1\epsilon_2(c_2 + \epsilon_1 d_2)^2 x^3 \mod x^4. \end{aligned}$$

The lemma follows directly from this equation. ■

We remark that the two cosets of $I_3^{\infty +}$ in I_3^∞ are in fact parabolas.

Lemma 17.
(1) For each $m \in \mathbb{N}$ and $a \in F$, there is an involution with series beginning $-x - ax^{2m} - ma^2x^{4m-1} + \cdots$.
(2) Thus there is a product of two proper involutions conjugate to $x + ax^{2m} + ma^2x^{4m-1}$.

Proof. In fact, the linear fractional transformations

$$\frac{x}{ax-1} = -x - ax^2 - a^2x^3 - \cdots$$

are involutive, for each $a \in F$. Taking $p = 2m-1$, replacing a by ap, and 'conjugating with x^p' gives

$$\frac{-x}{(1-apx^p)^{1/p}} = -x - ax^{2m} - ma^2x^{4m-1} + \cdots$$

so this is formally involutive. This proves (1).

(Note that this argument does not require the field F to have p-th roots of all its elements. When $F = \mathbb{R}$, and a is real, the left-hand-side is an involutive *function* (since p is odd), and is represented by the series near 0. Hence the right-hand-side is a formally-involutive series. This fact will not change when a becomes an element of an arbitrary field, since it amounts to a collection of polynomial identities.)

Following the series from (1) with $-x$, and applying Lemma 11, we get (2). ∎

We remark that there is a way to generate all involutions (cf. [7]), which allows one to make examples of the above kind, at will. For instance, to produce τ with the form $-x - x^4 \mod x^5$, one may take the unique solution to

$$\tau(x)^2 + \tau(x)^5 = x^2 + x^5$$

that has multiplier -1.

We also note that there are no polynomial involutions except $\pm x$, but we now see that the composition of two involutions may be a polynomial, and that any degree except 2 is possible.

Corollary 18. *Let $\phi \in H^+$ have conjugation-invariants $p = 2m - 1$ and $\alpha = m$ for some $m \in \mathbb{N}$. Then ϕ is the product of two proper involutions.*

Proof. From Lemma 17, ϕ is conjugate to a product of involutions. But conjugation preserves $I^{\circ 2}$, and propriety (the property of being proper). ∎

Lemma 19. *Let $\phi = x + \sum_{n=1}^{\infty} a_n x^n$ be the product $\tau_1 \circ \tau_2$ of two distinct proper involutions, and have conjugation-invariants $p = 2m - 1$ and α. Then $\alpha = m$.*

Proof. We may conjugate τ_1 to $-x$, and then for some m' we have

$$\begin{aligned} \tau_2 \quad = \quad & -x + a_{2m'} x^{2m'} + a_{2m'+2} x^{2m'+2} + \cdots \\ & + a_{4m'-2} x^{4m'-2} - m' a_{2m'}^2 x^{4m'-1} + \cdots , \end{aligned}$$

where $a_{2m'} \neq 0$, by Lemma 14. Thus $m' = m$ and $\alpha = m$, by Lemma 11. ∎

At this stage, we have completed the proof of assertion (1) of the main theorem. Note that, in particular, if $\phi \mod x^4$ belongs to $I_3^{\circ 2}$ but is not $x \mod x^4$, then $\phi \in I^{\circ 2}$ (because in those circumstances its invariants are $p = 1$ and $\alpha = 1$).

4.1. Proof of assertion (2). Let $\phi \in G$ and $\phi = -x$ modulo x^4. Thus

$$\phi = -x + \sum_{n=4}^{\infty} a_n x^n,$$

for some $a_n \in F$.

Take $\tau_1 = \sum_{n=1}^{\infty} -x^n$. Then τ_1 is an involution and obviously

$$\tau_1 \circ \phi = x - x^2 + x^3 + \sum_{n=4}^{\infty} b_n x^n$$

for some $b_n \in F$. Thus by the above note, $\tau_1 \circ \phi$ is a product $\tau_2 \circ \tau_3$ of involutions, hence $\phi = \tau_1 \circ \tau_2 \circ \tau_3$.

Now let ψ be any product of an odd number of proper involutions. Then $\psi \mod x^4$ belongs to I_3 (Lemma 16), and hence $\psi \mod x^4$ is conjugate in G_3 to $-x \mod x^4$. Thus there is some cubic $\lambda \in G$ such that $\lambda^{\circ -1} \circ \psi \circ \lambda = -x$ modulo x^4. Thus $\lambda^{\circ -1} \circ \psi \circ \lambda$ is the product of three involutions, and hence so is ψ.

The rest follows from Lemma 16.

4.2. Proof of assertion (3). From (2), it is clear that any product of proper involutions is the product of three or four. Thus $I^{\infty} = I^{\circ 4}$.

Obviously $I^{\infty} \subset \{\phi : \phi \mod x^4 \in I_3^{\infty}\}$.

To see the converse, let $\phi \mod x^4 \in I_3^{\infty}$.

If the multiplier is -1, we have just seen that ϕ is the product of three involutions.

If the multiplier is $+1$, then $-\phi$ has multiplier -1 and $-\phi \mod x^4$ belongs to I_3^{∞}, so $-\phi$ is the product of three involutions, and hence ϕ is the product of four.

The rest follows from Lemma 16.

4.3. Proof of assertion (4). From (3) we obtain immediately that $I_k^{\infty} = I_4^{\circ 4}$ for each $k \in \mathbb{N}$.

Suppose $k \le 6$, and $\phi \mod x^{k+1} \in I_k^{\infty +}$.

If $\phi \ne x \mod x^3$, then $\phi \in I^{\circ 2}$, so $\phi \mod x^{k+1} \in I_k^{\circ 2}$.

If $\phi = x \mod x^3$, but $\phi \ne x \mod x^5$, then one finds that $\phi = x + a_4 x^4 + a_6 x^6 \mod x^{k+1}$, with nonzero a_4, so changing the coefficient of x^7 to $2a_4^2$ (which does not affect $\phi \mod x^{k+1}$), we may arrange that ϕ is the product of two involutions, and again we are done.

If $\phi = x \mod x^5$, but $\phi \ne x \mod x^{k+1}$, then $k = 6$ and one finds that $\phi = x + a_6 x^6 \mod x^7$, with nonzero a_6, so changing the coefficient of x^{11} to $3a_6^2$ we may arrange that ϕ is the product of two involutions, and again we are done.

The remaining case is $\phi = x \mod x^{k+1}$, and in that case $\phi = (-x) \circ (-x)$.

One readily deduces that for $k \le 6$ each $\phi \in -I_k^{\infty +}$ is the product of three proper involutions. We have already seen that for $k = 3$, $-I_k^{\infty +}$ consists simply of involutions. When $k < 3$, this is also clear. Finally, one sees that $\tau^{\circ 2} = x \mod x^4$ if and only if $\tau^{\circ 2} = x \mod x^5$, and it follows that $-I_4^{\infty +}$ also consists entirely of involutions.

This concludes the proof.

5. Further Remarks on Products of Involutions

5.1. G_k has a faithful k-dimensional matrix representation, given by right-composition on the semigroup M, viewed as a vector space over F. From this point of view, the involutions are special reflections. In fact, there is a distinguished family of pairs (U,V), where U is a subspace of M of dimension floor$(k/2)$ and V is a complementary subspace, of dimension floor$(k+1)/2$, and each involution takes the form

$$\begin{cases} M = U \oplus V & \to & M \\ (x,y) & \mapsto & (x,-y) \end{cases}$$

The pairs (U,V) are of the form $(U_0 \circ g, V_0 \circ g)$, where U_0 is the space of even series, and V_0 is the space of odd series, and g ranges over G_k (or just H_k^+).

5.2. Dihedral groups of maps have proven useful in varied situations: that is why we embarked on this study. Two characteristics are particularly useful: (1) they are amenable, and hence susceptible to ergodic-theoretic methods, and (2) the maps admit time-reversal symmetries, and hence their dynamics is 'relatively simple'. It has been found useful if the discrete group $\langle\phi\rangle$ generated by a bijective map may be embedded in a one-parameter group.

In the present case, a power series $\phi \in H^+$ may be regarded as acting by right-composition on H^+, or on any H_k^+, and hence as a dynamical system. Note that H_k^+ may be regarded as the vector space F^{k-1} (equipped with a certain nonabelian group structure).

Consider the case $F = \mathbb{R}$ or $F = \mathbb{C}$. Then H_k^+ is a connected Lie group, hence the exponential map is surjective, and each $\phi \in H_k^+$ may be embedded in a 1-parameter subgroup $\{\exp ta\}$, with $\exp a = \phi$. Our result above allows us to obtain this conclusion for general F, provide $\phi \in I^{\circ 2}$, and also to obtain an embedding with a special property:

Corollary 20. *Let $\phi \in I^{\infty+}$ (resp. $I_k^{\infty+}$ for some $k \geq 2$) be the composition of two involutions. Then there exists a group homomorphism $\tau : (F,+) \to I^{\infty+}$ (resp. $\to I_k^{\infty+}$) such that $\tau(1) = \phi$ and each $\tau(t)$ is the composition of two involutions, $\tau_1(t)$ and $\tau_2(t)$. Moreover, $\tau_1(t)$ may be chosen independent of t.*

Proof. The case $\phi = x$ is trivial. Otherwise, ϕ is the composition of two proper involutions. Combining the main theorem with the idea in the proof of Lemma 17, we deduce that ϕ is conjugate to the formal series

$$\psi = \frac{x}{(1 - apx^p)^{1/p}}$$

for some odd $p \in \mathbb{N}$ and $a \in F^\times$. The map

$$\sigma : \begin{cases} F & \to & H^+ \\ t & \mapsto & \dfrac{x}{(1 - aptx^p)^{1/p}} \end{cases}$$

is a group homomorphism, and $\sigma(1)=\psi$. Moreover, each $\sigma(t)$ is the composition of $-x$ and another involution. So conjugating back, we get the desired τ. ■

6. Proof of Theorem 5

6.1. (b) ⇔ (c). Expanding the series in (b), we get

$$\frac{\mu x}{(1+\lambda x^p)^{1/p}} = \mu x + a x^{p+1} + b x^{2p+1} + \cdots,$$

where $a=\lambda/p$ and $b=\mu(p+1)a^2/2$. Thus the canonical conjugate polynomial form of this series is as in (c).

Hence (b) and (c) are equivalent.

It remains to show that (a) implies (b) or (c). This is the main part of the proof.

6.2. Preliminary Lemmas.

Lemma 21. *Let*

$$f = x + a x^{p+1} + \cdots,$$

with $a\neq 0$. Then f has infinite order.

Proof. Calculation yields

$$f^{\circ n} = x + n a x^{p+1} + \cdots \neq x.$$

■

Lemma 22. *Suppose that $f\in G$ has finite order $m\in\mathbb{N}$. Then f is conjugate in G to ωx, where ω is a primitive m-th root of unity in F.*

Proof. Let $f=\omega x+\sum_{j=2}^{+\infty} f_j x^j$. Then $x=f^{\circ m}=\omega^m x+\cdots$, so $\omega^m=1$.

If $\omega^n=1$, then $f^{\circ n}=x$, for otherwise $f^{\circ n}=x+\cdots$ has infinite order (by the previous lemma). Thus $\omega^n=1$ implies $m|n$, so ω is a primitive m-th root of 1.

Let

$$f = \omega x + a x^{p+1} \mod x^{p+2},$$

with $a\neq 0$.

We calculate that

$$x = f^{\circ m} = X + \left(1+\omega^p+\omega^{2p}+\cdots+\omega^{(m-1)p}\right)\omega^{m-1} a x^{p+1} + \cdots,$$

hence

$$1+\omega^p+\omega^{2p}+\cdots+\omega^{(m-1)p}=0,$$

hence m does not divide p.

Now consider

$$\left(x+\alpha x^{p+1}\right)^{\circ -1}\circ f\circ\left(X+\alpha x^{p+1}\right).$$

Calculation shows that, modulo x^{p+2}, this equals

$$\omega x + \{a + \alpha\omega(1-\omega^p)\}\, x^{p+1}.$$

Thus, choosing

$$\alpha = \alpha_1 = \frac{-a}{\omega(1-\omega^p)},$$

we obtain a series $f_1 = \omega x + a_1 x^{p_1} + \cdots$, conjugated to f by $g_1 = x + \alpha_1 x^{p+1}$, and having $p_1 > p$.

Proceeding inductively, we obtain a series f_{n+1} from f_n, by conjugating with a suitable $g_{n+1} = x + \alpha_{n+1} x^{p_n+1}$, and having $p_{n+1} > p_n$.

Thus, letting $g = \prod_{n=1}^{\infty} g_n$, we get $g^{\circ -1} \circ f \circ g = \omega x$. ■

6.3. Proof that (a) implies (c). For $p \in \mathbb{N}$, let us denote the multiplicative group of p-th roots of unity in F by S_p. This is a cyclic group whose order divides p.

We consider first $f \in R(G)$ with multiplier $+1$, and let $p = p(f)$, i.e.
$f = x + ax^{p+1} \bmod x^{p+2}$, with $a \neq 0$.

Let h reverse f. Then $h^{\circ 2} \in C_f$ (and $h \notin C_f$, since f is not involutive).

The centralisers of elements of G are understood, and an account may be found, for instance, in Lubin's paper [11, Proposition 1.4]. We may summarise the result as follows:

Proposition 23. *The multiplier map μ is a homomorphism from G onto $F^\times$, mapping C_f into S_p, and such that $C_f \cap \ker\mu \le J_p$. The map*

$$\Phi : \left\{ \begin{array}{rcl} C_f & \to & S_p \times F, \\ \lambda x + bx^{p+1} & \mapsto & (\lambda, b/\lambda), \end{array} \right. \tag{1}$$

is a group homomorphism. In particular, C_f is abelian.

In this statement, we understand that $S_p \times F$ has the cartesian product group structure induced by the multiplicative structure on S_p and the additive structure on F.

Let us endow the product $S_{2p} \times F$ with the twisted product defined by setting

$$(\lambda, c) \cdot (\lambda', c') = (\lambda\lambda', \lambda^p c + c'),$$

whenever $\lambda, \lambda' \in S_{2p}$ and $c, c' \in F$. This works because $\lambda \mapsto \lambda^p$ is a homomorphism from S_{2p} onto $S_2 = \{\pm 1\}$.

Observe that in this (nonabelian) group the element $(\omega, 0)$ has order $2p$, and reverses each element $(1, c)$.

We define an extension of Φ from C_f to $C_f \cup R_f$ by using the same formula (1), and see by routine calculation that it is still a group homomorphism when extended, and hence is an isomorphism.

Define $\tau=\Phi^{-1}(\omega,0)$. Then τ is an element of G of order $2p$, and reverses f.

By Lemma 22, we may choose $g\in G$ that conjugates τ to ωx.

Replacing f by its conjugate $g^{\circ-1}\circ f\circ g$, we may now assume that f is reversed by ωx.

Since f commutes with $\omega^2 x$, it takes the form

$$f=x+\sum_{j=1}^{\infty} f_j x^{jp+1},$$

and the statement that ωx reverses f amounts to the identity:

$$f^{\circ-1}=x+\sum_{j=1}^{\infty}(-1)^j f_j x^{jp+1}.$$

Substituting f for x in this, we get:

$$\begin{array}{rcl} x &=& x+\sum_{j=1}^{\infty} f_j x^{jp+1} \\ && +\sum_{j=1}^{\infty}(-1)^j f_j \left(x+\sum_{k=1}^{\infty} f_k x^{kp+1}\right)^{jp+1}. \end{array}$$

Subtracting x from both sides, cancelling x^{p+1}, and replacing x^p by y, we get, modulo y^2:

$$0=f_1+f_2y-f_1(1+f_1y)^{p+1}+f_2y(1)^{2p+1},$$

so that

$$f_2=\frac{(p+1)f_1^2}{2}.$$

Since the conjugacy class of f in G is determined by (p and) the coefficients f_1 and f_2, we conclude that f is conjugate to

$$x+f_1x^{p+1}+\frac{(p+1)f_1^2}{2}x^{2p+1}.$$

Thus condition (c) holds, and we are done, in case $\mu(f)=+1$.

Now suppose that $f\in R(G)$ has multiplier -1.

Let $h\in G$ reverse f.

Conjugating if need be, we may assume that f is an involution or takes the form

$$f=-x+ax^{p+1}+bx^{2p+1},$$

with $p\in\mathbb{N}$, even, and $a\neq 0$.

In the former case, f is conjugate to $-x$, and we are done. In the latter, $f^{\circ 2}$ is also reversed by h, and (modulo x^{2p+2}) we calculate that

$$f^{\circ 2}=x-2ax^{p+1}+\left\{(p+1)a^2-2b\right\}x^{2p+1}.$$

Thus, by Part (1),

$$\begin{aligned}(p+1)a^2-2b &= \frac{(p+1)(-2a)^2}{2},\\ b &= -\frac{(p+1)a^2}{2}.\end{aligned}$$

Thus f is conjugate to

$$-x+ax^{p+1}-\frac{(p+1)a^2}{2}x^{2p+1},$$

as required.

6.4. Proof of Corollary 6.

Proof.
(a) implies (b):
We saw in the course of the proof of Theorem 5 that (a) implies that ϕ may be reversed by a series, say τ, of finite even order $2p$. But since each odd power of an element that reverses ϕ also reverses ϕ, it follows that $\sigma=\tau^{\circ k}$ also reverses ϕ, where k is the largest odd number dividing p. Now the order of σ is a power of 2, and the remainder of (b) follows from Lemma 22.

(b) implies (c):
Suppose that g reverses ϕ and has finite order. Let $h=g\circ\phi$. Then $\phi=g^{\circ-1}\circ h$, and

$$h^{\circ 2}=g\circ(\phi\circ g)\circ\phi=g\circ g\circ\phi^{\circ-1}\circ\phi=g^{\circ 2},$$

so (c) holds.

(c) implies (a):
Condition (c) implies that

$$g^{\circ-1}\circ\phi\circ g=g^{\circ-2}\circ h\circ g=h^{\circ-2}\circ h\circ g=h^{\circ-1}\circ g=\phi^{\circ-1}.$$

■

By way of example, the series

$$\frac{\omega x}{(1+\lambda x^p)^{1/p}}$$

has order $2p$, whenever $\omega^p=-1$, so in that case the series

$$\frac{x}{(1+\lambda x^p)^{1/p}}$$

can be reversed by an element of order $2p$, but not by any element of order p. Of course, the field F may or may not have such an element ω.

6.5. Proof of Corollary 7. The "if" direction is trivial.

Regarding the other direction, we saw in the course of proving Theorem 5 that when $\phi \in R$ has conjugation-invariant $p = p(\phi)$, then it can be reversed by a series of order $2p$. Thus when p is odd, ϕ can be reversed by an involution, and hence is the composition of two involutions.

6.6. Proof of Corollary 8. We have seen that if a reversible series has conjugacy invariant p, then the multiplier of any reversing series is a primitive $2p$-th root of unity in F. Hence, if there are no primitive fourth roots of unity, it follows that each reversible series has an odd invariant p, and hence is the composition of two involutions.

Conversely, if ω is a primitive fourth root of 1, then the series (expansion of) $x/\left(1+x^2\right)^{1/2}$ belongs to R, by the Theorem, but not to $I^{\circ 2}$, so $R \neq I^{\circ 2}$.

7. Proof of Theorem 9

7.1. Proof of (1). By Corollary 8 we have $R = I^{\circ 2}$, and the rest of assertion (1) follows from Theorem 4.

7.2. Proof of (2). Suppose that F has a square root of -1, and fix $f \in H$. Take $\mu = \mu(f)$ and $p = p(f)$. Then up to conjugacy,

$$f = \mu x + a x^{p+1} + b x^{2p+1} \mod x^{2p+2},$$

where p is even if $\mu = -1$.

1°. Consider first the case when $p = 1$.

By Theorem 5, the series

$$\begin{array}{rcl} g & = & \mu x + a x^2 + \mu a^2 x^3 \\ \text{and} & & \\ h & = & x + c x^3 + \frac{3c^2}{2} x^5 \end{array}$$

belong to R, so by suitable choice of c we get

$$g \circ h = f \mod x^4,$$

so that $f \in R^{\circ 2}$.

2°. Next, consider the case in which $p \geq 2$.

In this case, $f = \pm x \mod x^4$, so by Theorem 4 $f \in I^{\circ 4} \subset R^{\circ 2}$.

3°. Finally, consider the case in which $p = 2$, so $f = \mu x + a x^3 + b x^5 \mod x^6$.

Take $g = \mu x + a x^3 + x^4 + c x^5$. We can choose $c \in F$ to make $g \in R$, by Lemma 11, part (2), and Theorem 5. Thus

$$g^{\circ -1} \circ f = x - \mu x^4 + d x^5 \mod x^6,$$

for some $d \in F$. Applying Lemma 11 and Theorem 5 again, we may choose $e \in F$ such that

$$h = x - \mu x^4 + dx^5 + ex^7$$

belongs to R. Then $h^{\circ -1} \circ g^{\circ -1} \circ f = x \mod x^6$, so $f = g \circ h \mod x^6$, so f is conjugate to $g \circ h$, and so $f \in R^{\circ 2}$.

Thus we have shown that in all cases $R^{\circ 2} = H$, and the rest of the assertions follow from Theorem 4.

8. Concluding Remarks

8.1. Kasner's Work. Kasner [9] worked over the field of complex numbers $\mathbb{C}$, with the larger group $G \cup \bar{G}$ of series

$$f = \sum_{n=1}^{\infty} f_n z^n \text{ or } f = \sum_{n=1}^{\infty} f_n \bar{z}^n,$$

and he focussed on the generation of series having unimodular multipliers by involutions in the large group. This group has, as well as our proper involutions (conjugate to $-z$), the "anticonformal reflections" (conjugate to $\bar{z}$). With this richer collection of involutions, one can generate all series having unimodular multipliers (not just ± 1), using just 4 involutions.

He also asserted his results for the "complex plane". One may, in fact, consider Kasner's group $K = G \cup \bar{G}$ over an arbitrary field F of characteristic 0, by regarding $\bar{x}$ as a formal object, and defining the involutive action $f \mapsto \bar{f}$ on K by

$$\begin{array}{ccc} \sum_{n=1}^{\infty} c_n x^n & \mapsto & \sum_{n=1}^{\infty} c_n \bar{x}^n, \\ \sum_{n=1}^{\infty} c_n \bar{x}^n & \mapsto & \sum_{n=1}^{\infty} c_n x^n, \end{array}$$

i.e. one conjugates only the symbol x or $\bar{x}$, but not the coefficient. We believe that in case $F = \mathbb{C}$, this is what Kasner had in mind by series in the "complex plane".

For general F, the group K has, as before, the involutions that belong to G, and the "anticonformal reflections", the conjugates of the series $\bar{x}$. Each series $f \in G$ for which $\mu(f)$ is not a root of unity in F is conjugate to $\mu(f)x$, and hence is the product of two conformal reflections. Given this observation, one can recover the rest of the results in Kasner's paper from the main results stated above.

8.2. Smooth Germs in One Variable. Let $\mathcal{D}$ be the group of germs of C^∞ diffeomorphisms fixing $0 \in \mathbb{R}$. The (truncated) Taylor series map

$$T : f \mapsto \sum_{n=1}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

is a group homomorphism from $\mathcal{D}$ onto $G(\mathbb{R})$ (— the surjectivity is a celebrated result of É. Borel). Reversible germs map to reversible series, so a germ is reversible only if its Taylor series is reversible. Now Sternberg and Mather established that if $Tf \neq e$, then Tf determines the conjugacy class of f in $\mathcal{D}$. So it follows by combining this result with Borel's that a germ f having $Tf \neq e$ is reversible in $\mathcal{D}$ if and only if Tf is reversible in $G(\mathbb{R})$.

The author and Maria Roginskaya have studied conjugacy in $\mathcal{D}$, and identified the reversible germs having $Tf = e$, but this is a different story and the details will appear elsewhere.

8.3. Further problems. One may consider formally-invertible series over any commutative ring A with identity, and there is a good deal to be done to answer the corresponding problems in that generality. Lubin [11] provides much of the necessary foundation for this study in case A is an integral domain of characteristic zero. The questions for the Nottingham groups (in which A is a finite field) appear to be unexplored so far, although the elements of finite order are well-understood (at least when the order of A is odd), thanks to Klopsch [3, p. 16]. The corresponding problems for formal series in more than one variable, and for convergent series over a general complete metric field also appear to be open.

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