Algebras of smooth functions

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Abstract. There is as yet no satisfactory description of the closed subalgebras of infinitely-differentiable real-valued functions on a smooth manifold. The same is true of the algebra of C^k functions, for $k \geq 1$. The Stone–Weierstrass Theorem solves the problem for C^0 functions. Whitney's Spectral Theorem provides a description of the closed *ideals* in the general case. Nachbin described the *maximal* closed subalgebras of C^k in 1949, and he responded to a question of Segal by proposing a conjecture about the general case. We describe some further progress on the problem, and a refinement of Nachbin's conjecture.

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1. Introduction

(1.1) Let $\psi_1, \ldots, \psi_r \in C^\infty(\mathbb{R}^d, \mathbb{R})$ and let $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$. When do there exist polynomials $p_n \in \mathbb{R}[x_1, \ldots, x_r]$ such that

$$p_n(\psi_1(x), \ldots, \psi_r(x)) \to f(x)$$

in the usual topology on $C^\infty(\mathbb{R}^d, \mathbb{R})$? For instance, which $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$ may be obtained as

$$\lim_{n\to\infty} p_n(x^3, \cos x)$$

where the p_n are real polynomials in two variables?

One could also ask about C^k approximation, for an integer $k \geq 0$, and other variations.

In the case of C^0 approximation, that is, locally uniform approximation, the Stone–Weierstrass theorem provides all the answers. It allows us to answer three different problems about a given unital algebra A of continuous functions:

1. When is $\operatorname{clos} A = C^0(X)$?
2. When does $f \in \operatorname{clos} A$?
3. When is A closed?

For a C^k manifold M, one has similar problems about subalgebras of $C^k(M)$, but they are not so simple, nor do they reduce in this way to one problem.

We concentrate on the case $k = \infty$, which is typical of C^k, $k \geq 1$.

(1.2) Let M be a connected C^∞ manifold, and give $C^\infty(M, \mathbb{R}^r)$ its usual Fréchet space topology. Then $C^\infty(M) = C^\infty(M, \mathbb{R})$ is a Fréchet algebra under pointwise operations. Let A be a subalgebra of $C^\infty(M)$, containing the constants. The main problem we consider is to describe the closure of A in $C^\infty(M)$.

Briefly, the history of the problem is as follows.

Whitney [W] described the closed ideals. The problem is local, and closed ideals are determined precisely by the point evaluations, point derivations and higher point derivations that they annihilate.

Nachbin [N1] described the maximal closed subalgebras. They are all of codimension 1, and apart from the maximal ideals one has the algebras

$$\{f \in C^\infty : f(a) = f(b)\}$$

where $a, b \in M$ with $a \neq b$, and the algebras

$$\{f \in C^\infty : \partial f = 0\}$$

where $\partial \in TM$ is a (first-order) bounded point derivation.

Segal asked in 1949 for a description of clos A, in general. Nachbin responded with this ***conjecture***:

> $f \in \operatorname{clos} A$ if and only if for each compact subset K of a coordinate patch such that all functions of A are constant on K, and for each $\epsilon > 0$ and $k \in \mathbb{N}$, there exists $p \in A$ such that
>
> $$|\partial^i p - \partial^i f| < \epsilon \text{ on } K, \quad \forall |i| \leq k,$$
>
> where ∂^i denotes the partial derivative corresponding to the multiindex $i \in \mathbb{Z}^d_+$.

It is of course possible to rephrase this in coordinate-free terms.

(1.3) We focus on the special case in which $M = \mathbb{R}^d$ and A is finitely generated. These restrictions are of no great consequence.

So let $\Psi = (\psi_1, \ldots, \psi_r) \in C^\infty(\mathbb{R}^d, \mathbb{R})$, and denote

$$\mathbb{R}[\Psi] = \mathbb{R}[\psi_1, \ldots, \psi_r], \quad A(\Psi) = \operatorname{clos}_{C^\infty(\mathbb{R}^d)} \mathbb{R}[\Psi],$$

and

$$C^\infty(\Psi) = \{g \circ \Psi : g \in C^\infty(\mathbb{R}^r, \mathbb{R})\}.$$

Lemma 1. *With the above notation, we have* $A(\Psi) = \operatorname{clos}_{C^\infty} C^\infty(\Psi)$.

Proof. This follows at once from the fact that $\mathbb{R}[x_1, \ldots, x_r]$ is dense in $C^\infty(\mathbb{R}^r, \mathbb{R})$ in the C^∞ topology. □

So we are interested in the functions f that may be approximated by compositions $g_n \circ \Psi$, with $g_n \in C^\infty(\mathbb{R}^r, \mathbb{R})$. Consider a function f that is exactly equal to $g \circ \Psi$ for some $g \in C^\infty(\mathbb{R}^r, \mathbb{R})$. Denoting the Taylor series at a of f by $T_a f$ and the Taylor series minus its constant term by $T'_a f$, so that $T'_a = T_a f - f(a)$, we have the chain rule

$$T_a(g \circ \Psi) = (T_{\Psi(a)}g) \circ (T'_a\Psi)$$

where $\circ$ denotes the formal composition of power series. Thus $f \in \mathbb{R}[[T'_a\Psi]]$, where for $p \in \mathbb{R}[[x_1, \dots, x_d]]^r$ an r-tuple of formal power series with $p(0) = 0$ (that is, no constant term) we denote

$$\mathbb{R}[[p]] = \{q \circ p : q \in \mathbb{R}[[x_1, \dots, x_r]]\}.$$

The algebra $\mathbb{R}[[x_1, \dots, x_d]]$ of formal power series has a standard Fréchet algebra topology, and the map

$$T_a : \begin{cases} C^\infty(\mathbb{R}^d) \to \mathbb{R}[[x_1, \dots, x_d]] \\ f \mapsto T_a f \end{cases}$$

is continuous.

Lemma 2. *Let* $p \in \mathbb{R}[[x_1, \dots, x_d]]^r$, *with* $p(0) = 0$. *Then* $\mathbb{R}[[p]]$ *is closed in* $\mathbb{R}[[x_1, \dots, x_d]]$.

We will shortly prove a strengthened form of this lemma, so we omit the proof.

Corollary 3. *We have* $A(\Psi) \subset \bigcap_{a \in \mathbb{R}^d} \{f : T_a f \in \mathbb{R}[[T'_a\Psi]]\}$. □

When Ψ fails to be injective, there is a tighter restriction on the elements of $A(\Psi)$. First, we give the stronger version of Lemma 2.

Lemma 4. *Suppose* $\{p_\alpha : \alpha \in I\}$ *is a family of elements of* $\mathbb{R}[[x_1, \dots, x_d]]^r$ *having* $p_\alpha(0) = 0, \forall \alpha \in I$. *Suppose* $q_n \in \mathbb{R}[[x_1, \dots, x_r]]$ $(n = 1, 2, 3, \dots)$ *and* $q_n \circ p_\alpha \to t_\alpha$ *in* $\mathbb{R}[[x_1, \dots, x_d]]$ *as* $n \to \infty$, *for each* $\alpha \in I$. *Then there exists* $f \in \mathbb{R}[[x_1, \dots, x_r]]$ *such that* $t_\alpha = f \circ p_\alpha$, $\forall \alpha \in I$.

Proof. Let p_α, q_n, t_α be as in the statement of the lemma.

Observe that $q_n(0) = q_n(p_\alpha(0)) \to t_\alpha(0)$, so all t_α share the same constant term. Let f_0 be this term.

We proceed by induction. Our induction hypothesis P_k is that f_k belonging to $\mathbb{R}[x_1, \dots, x_r]_k$ (the space of polynomials in r variables of degree at most k) may be chosen so that the following three properties hold:

$$(1) \qquad T^k t_\alpha = T^k(f_k \circ p_\alpha), \quad \forall \alpha \in I,$$

$$(2) \qquad \forall\, j > k \quad \exists \quad q_j \in \mathbb{R}[x_1, \dots, x_r]_j \quad \text{such that}$$

$$(*) \quad \begin{cases} T^j t_\alpha = T^j(q_j \circ p_\alpha), \quad \forall \alpha \in I, \\ T^k q_j = f_k, \end{cases}$$

and

(3) $$T^j f_k = f_j \quad \text{for } 0 \le j \le k.$$

First we verify P_0. Properties (1) and (3) are trivial. To verify (2), we fix $j > 0$ and begin by defining

$$S_\alpha = \{q \in \mathbb{R}[x_1, \ldots, x_r]_j : T^j t_\alpha = T^j(q \circ p_\alpha)\}, \quad \forall \alpha \in I.$$

Then each S_α is an affine subspace of $\mathbb{R}[x_1, \ldots, x_r]_j$, and hence is closed and finite dimensional, or perhaps empty. In fact, S_α is never empty. More is true.

Claim. Each finite intersection $S_{\alpha_1} \cap \cdots \cap S_{\alpha_n}$ is nonempty.

Proof of Claim. Fix $\alpha_1, \ldots, \alpha_n \in I$. Consider the linear space

$$V = \{(T^j(q \circ p_{\alpha_1}), \ldots, T^j(q \circ p_{\alpha_n})) : q \in \mathbb{R}[x_1, \ldots, x_r]_j\}.$$

This V is a linear subspace of the finite-dimensional vector space $(\mathbb{R}[x_1, \ldots, x_d]_j)^n$, and hence is closed. Thus $(T^j t_{\alpha_1}, \ldots, T^j t_{\alpha_n}) \in V$, so there exists $q \in \mathbb{R}[x_1, \ldots, x_r]_j$ with $T^j(q \circ p_{\alpha_i}) = T^j t_{\alpha_i}$ for $i = 1, \ldots, n$.

This proves the claim.

Thus each intersection $\bigcap_{i=1}^n S_{\alpha_i}$ is a nonempty finite-dimensional affine subspace of $\mathbb{R}[x_1, \ldots, x_r]_j$. Thus there exist $\alpha_1, \ldots, \alpha_n \in I$ such that $\dim \bigcap_{i=1}^n S_{\alpha_i}$ is minimal, hence

$$\bigcap_{\alpha \in I} S_\alpha = \bigcap_{i=1}^n S_{\alpha_i} \ne \emptyset.$$

Pick $q_j \in \bigcap_{\alpha \in I} S_\alpha$. Then $q_j(0) = t_\alpha(0) = f_0$. Thus $(*)$ holds (for $k = 0$), and we have completed the verification of P_0.

Now suppose $f_0, \ldots, f_k$ have been chosen, and P_k holds.

For $j > k + 1$ let

$$A_{k+1,j} = \{T^{k+1} q_j : (*) \text{ holds}\}.$$

Then $A_{k+1,j}$ is a nonempty affine subset of $\mathbb{R}[x_1, \ldots, x_r]_{k+1}$ and

$$A_{k+1,j+1} \subset A_{k+1,j}, \quad \forall j > k + 1.$$

Thus

$$\dim A_{k+1,j+1} \le \dim A_{k+1,j},$$

so there exists $J > k + 1$ such that $A_{k+1,j} = A_{k+1,J}$, $\forall j > J$. Thus

$$\bigcap_{j=k+2}^{+\infty} A_{k+1,j} = A_{k+1,J}$$

is nonempty. Pick $f_{k+1} \in A_{k+1,J}$. Clearly f_{k+1} enjoys properties (1), (2) and (3), with k replaced by $k + 1$.

By induction, we obtain f_k for all $k \in \mathbb{N}$, and, by property (3), $\{f_k\}$ converges in $\mathbb{R}[[x_1, \ldots, x_r]]$ to a power series, f, such that $T^k f = f_k \ \forall k \geq 0$. By property (1), $T^k t_\alpha = T^k(f \circ p_\alpha)$, $\forall k$, $\forall \alpha$, so $t_\alpha = f \circ p_\alpha$, $\forall \alpha \in I$. □

Corollary 5. *Let $f \in A(\Psi)$. Then for each value $b \in \operatorname{im}\Psi$ there exist $q_b \in \mathbb{R}[[x_1, \ldots, x_r]]$ such that*

$$T_a f = q_b \circ T'_a \Psi$$

whenever $\Psi(a) = b$.

This allows us to see that the following sharper statement is equivalent to Nachbin's conjecture:

Conjecture (f.g. case):

$$A(\Psi) = \bigcap_{b \in \operatorname{im}\Psi} \{f \in C^\infty(\mathbb{R}^d) \mid \exists q \in \mathbb{R}[[x_1, \ldots, x_r]]: T_a f = q \circ T'_a \Psi, \forall a \in \Psi^{-1}(b)\}.$$

One may also formulate a similar version of the conjecture in the general case, in terms of power series in an arbitrary number of indeterminates.

(1.4) Now *we assume that Ψ is injective* (that is, $\mathbb{R}[\Psi]$ is separating). Then the conjecture simplifies to the formula:

$$A(\Psi) = \bigcap_{a \in \mathbb{R}^d} T_a^{-1}\mathbb{R}[[T'_a \Psi]].$$

In view of the fact that $T_a^{-1}\mathbb{R}[[T'_a\Psi]] = \mathbb{R}[[x_1, \ldots, x_d]]$ when $\operatorname{rank} D\Psi(a) = d$, we may restate this conjecture in the form

$$(*) \qquad A(\Psi) = \bigcap_{a \in \operatorname{crit}\Psi} T_a^{-1}\mathbb{R}[[T'_a \Psi]].$$

Here we have denoted the set of critical points of Ψ by $\operatorname{crit}\Psi$. Note that since Ψ is injective we necessarily have $r \geq d$, so $\operatorname{crit}\Psi = \{a : \operatorname{rank} D\Psi(a) < d\}$.

Progress to date on this conjecture is as follows.

Tougeron's Theorem (1971) [T1]. *Let $\Psi \in C^\infty(\mathbb{R}^d, \mathbb{R}^r)$. Suppose that for each compact $K \subset \mathbb{R}^d$ there exist $\alpha > 0$ and $\beta > 0$ such that*

$$|\Psi(x) - \Psi(y)| \geq \alpha |x - y|^\beta, \quad \forall x, y \in K.$$

Then $(*)$ holds.

This applies in particular to all real-analytic Ψ, and all Ψ such that $\operatorname{crit}\Psi$ is discrete and consists entirely of critical points of finite order. Tougeron actually proved a more general result, which delivers the full Nachbin conjecture for real-analytic Ψ, even when Ψ is not injective. More recently, we have shown that the conjecture holds for all injective Ψ in one dimension.

Theorem A (1996) [A]. *Suppose* $\Psi \in C^\infty(\mathbb{R}, \mathbb{R}^r)$ *is injective. Then* $(*)$ *holds.*

The conjecture remains open in general, but we have made some progress on the case of maps that are 'at the opposite extreme' to those covered by Tougeron's result. First, we define f to be *flat at a* if $T_a f$ is a constant series, i.e., $T_a f = f(a)$.

Theorem B. *Suppose* $\Psi \in C^\infty(\mathbb{R}^d, \mathbb{R}^r)$ *is injective, and* $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$. *Suppose* f *is flat on* crit Ψ. *Then* $f \in A(\Psi)$.

As corollaries we obtain two more cases of the conjecture:

Corollary 6. *If* Ψ *is injective and is flat on* crit Ψ, *then* $(*)$ *holds.*

Corollary 7. *If* Ψ *is injective and* crit Ψ *is discrete, then* $(*)$ *holds.*

We will prove Theorem B below. Corollary 6 is immediate. Corollary 7 follows readily, in view of Borel's Theorem [T2], which states that

$$T_a C^\infty(\mathbb{R}^d) = \mathbb{R}[[x_1, \ldots, x_d]], \quad \forall a \in \mathbb{R}^d,$$

that is, each power series is the Taylor series of some smooth function. Given $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$ with

$$T_a f \in \bigcap_{a \in \operatorname{crit} \Psi} T_a^{-1} \mathbb{R}[[T_a' \Psi]],$$

Borel's Theorem allows us to find $g_a \in C^\infty(\mathbb{R}^r, \mathbb{R})$ with $f - g_a \circ \Psi$ flat at a, and since crit Ψ is discrete we may patch these g_a together to obtain a single g such that $f - g \circ \Psi$ is flat on crit Ψ, and hence, by theorem B, $f \in g \circ \Psi + A(\Psi) = A(\Psi)$.

2. The ingredients of the proofs

(2.1) The basic problem in all cases relates to the diagram in Figure 1.

Given a map $f : \mathbb{R}^d \to \mathbb{R}$ with the 'right kind' of Taylor series, we want to find $g_n : \mathbb{R}^r \to \mathbb{R}$ so that $g_n \circ \Psi \to f$ in C^∞. Tougeron's approach is based on a clever transfinite induction (with its origins in Malgrange's work) combined with the use of Whitney partitions on $\mathbb{R}^r$, in order to construct g_n directly. We call this a 'right-side approach', because it operates on the right-hand side of Figure 1, and we observe that such an approach has little chance of dealing with general Ψ, and in particular with places where Ψ is flat in some directions. So we have developed a 'left-side approach', precisely to deal with places where Ψ is flat. As it turns out, the left-side approach has more general application. The basic ingredient is the following simple 'factorisation lemma', from [A].

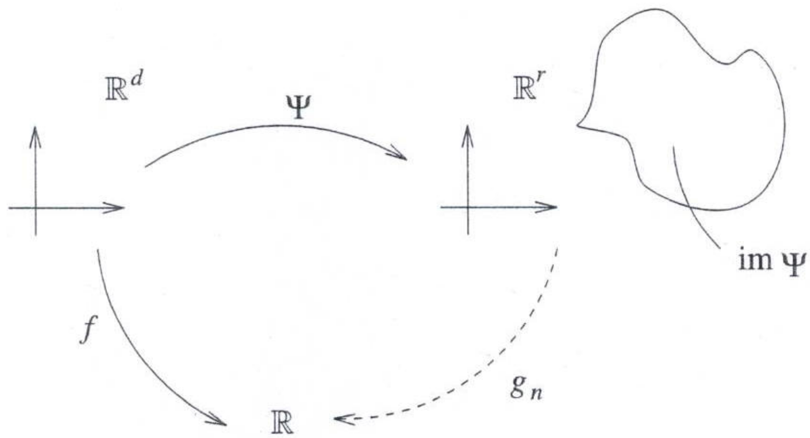

Figure 1

We say that a function $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$ is *locally-constant near a closed set E* if for each $a \in E$ there exists an open neighbourhood U of a such that f is constant on U.

Lemma 8 ([A], Lemma 7). *If* $\Psi \in C^\infty(\mathbb{R}^d, \mathbb{R}^r)$ *is injective,* $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$ *is locally-constant near* crit Ψ, *and* K *is a compact subset of* $\mathbb{R}^d$, *then there exists* $g \in C^\infty(\mathbb{R}^r, \mathbb{R})$ *such that* $g \circ \Psi = f$ *on* K.

This reduces the problem of proving Theorem B to that of approximating a function f that is flat on crit Ψ by a sequence of functions that are locally-constant near crit Ψ. To do this, we use some technical tricks: proxy distance, and the (η, E) equivalence relation. We now describe these.

(2.2) The distance function $x \mapsto \text{dist}(x, E)$, where $E \subset \mathbb{R}^d$, is a Lip 1 function, but is usually no smoother than that. It is convenient to have C^∞ 'substitutes' for $\text{dist}(x, E)$. So we make the following definition.

Definition. Let $E \subset \mathbb{R}^d$ be closed and $\kappa_m \geq 1 (m = 0, 1, 2, \ldots)$. A C^∞ function $d_E : \mathbb{R}^d \to [0, \infty)$ is called a $\{\kappa_m\}$ *proxy distance for E* if

$$\kappa_0^{-1} d_E(x) \leq \text{dist}(x, E) \leq \kappa_0 d_E(x), \quad \forall x \in \mathbb{R}^d,$$

and

$$|D^m d_E(x)| \leq \kappa_m \, \text{dist}(x, E)^{1-m}, \quad \forall x \in \mathbb{R}^d, \ \forall m \geq 1.$$

Lemma 9. *There is a sequence* $\{\kappa_m\}_{m=0}^{+\infty}$, *depending only on d, such that each nonempty closed set* $E \subset \mathbb{R}^d$ *has a* $\{\kappa_m\}$ *proxy distance.*

Proof. Fix $E \subset \mathbb{R}^d$, closed. Let $\{\phi_n\}_{n=1}^{\infty} \subset C^{\infty}(\mathbb{R}^d, \mathbb{R})$ be a standard Whitney partition of the identity on $\mathbb{R}^d \setminus E$. Thus

$$0 \le \phi_n \le 1 \text{ on } \mathbb{R}^d, \quad \sum \phi_n(x) = 1, \quad \forall x \in \mathbb{R}^d,$$

no more than N of the ϕ_n are non-zero at any one point, and letting $\beta_n = \operatorname{diam}(\operatorname{spt} \phi_n)$, we have

$$\mu^{-1}\beta_n \le \operatorname{dist}(x, E) \le \mu\beta_n, \quad \forall x \in \operatorname{spt} \phi_n,$$
$$|D^k \phi_n(x)| \le c_k \beta_n^{-k}, \quad \forall x \in \mathbb{R}^d, \quad \forall k \ge 1,$$

where the constants N, μ, and c_k $(k = 1, 2, \ldots)$ depend on d but are independent of E and n. See, for example, [T2]. Let $d_E(x) = \sum_{n=1}^{+\infty} \beta_n \phi_n(x)$. Then one readily verifies that d_E is a $\{\kappa_m\}_{m=0}^{\infty}$ proxy distance for E, where $\kappa_0 = \mu$ and $\kappa_m = N c_m \mu^{m-1}$, $\forall k \ge 1$. □

(2.3) Given a function flat on crit Ψ, we are going to approximate it by a function that is constant on various sets. In order to control its derivatives, we need to manage the separation of these various sets. This is where (η, E) equivalence comes in.

Definition. Let $E \subset \mathbb{R}^d$ and $\eta > 0$ be given. Let

$$N(E, \eta) = \{x \in \mathbb{R}^d : \operatorname{dist}(x, E) < \eta\}.$$

For $x, y \in N(E, \eta)$ we say that *x and y are (η, E) equivalent* if there exists a chain $B_1, \ldots, B_n$ of closed balls of $\mathbb{R}^d$ such that radius $B_i = \eta$, $B_i \cap E \neq \emptyset$, $B_i \cap B_{i+1} \neq \emptyset$, $x \in B_1$, and $y \in B_n$.

Lemma 10. (a) *(η, E) equivalence is an equivalence relation on $N(E, \eta)$.*
(b) *If $x, y \in N(E, \eta)$ are not (η, E) equivalent, then $|x - y| > \eta$.*
(c) *If E is bounded, then there are only a finite number of distinct (η, E) equivalence classes.*

Proof. This is easy to check. For (c), one obtains in fact that the number of equivalence classes is no greater than (for instance) $\left(\frac{4 \operatorname{diam} E}{\eta}\right)^d$. □

(2.4) Now we have a lemma about the variation of a sufficiently flat function on an (η, E) equivalence class. We say that $f \in C^{\infty}(\mathbb{R}^d, \mathbb{R})$ is *k-flat* at a point a if $T_a f$ has no terms of degree less than or equal to k, that is, $D^i f(a) = 0$, $\forall i \le k$. We let $\operatorname{var}_E f = \sup_E f - \inf_E f$.

Lemma 11. *Let $f \in C^{\infty}(\mathbb{R}^d, \mathbb{R})$, $R > 0$, $k \in \mathbb{N}$, $0 < \eta < R$, and*

$$M = \sup_{\mathbb{B}(0,R)} \max_{j \le k+d+1} |D^j f|.$$

Let $E \subset \mathbb{B}(0, R)$ be closed, f be $(k + d + 1)$-flat on E, $x, y \in N(E, \eta)$, and x be (η, E) equivalent to y. Then

$$|f(x) - f(y)| \leq 2^{d+1} M R^d \eta^{k+1}.$$

Proof. . Since they are (η, E) equivalent, x and y may be linked by a chain of balls $B_1, \ldots, B_n$ of radius η, each meeting E. Since f is flat on E, Taylor's formula with remainder yields

$$\operatorname{var}_{B_i} f \leq 2M\eta^{k+d+1}.$$

We may assume that each B_i meets only $B_{i\pm1}$, so n is at most twice the maximum number of disjoint balls of radius η one may fit into $\mathbb{B}(0, 2R)$, so $n \leq (2R/\eta)^d$. Thus

$$|f(x) - f(y)| \leq \sum_{i=1}^{n} \operatorname{var}_{B_i} f \leq 2^{d+1} M R^d \eta^{k+1},$$

as required. □

(2.5) We can now prove Theorem B. By Lemma 8, it suffices to prove the following.

Lemma 12. *Let $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$ be flat on the closed set $E \subset \mathbb{R}^d$. Then f is the C^∞ limit of a sequence of functions $f_n \in C^\infty(\mathbb{R}^d, \mathbb{R})$ such that f_n is locally constant near $E \cap \mathbb{B}(0, n)$.*

This in turn follows by a diagonal argument once we can show the following:

Lemma 13. *Let $f \in C^\infty(\mathbb{R}^d)$ be flat on the closed set E, and let $R > 1$ and $k \in \mathbb{N}$ be given. Then there exist $g_n \in C^\infty(\mathbb{R}^d)$, locally constant near $E_R = E \cap \mathbb{B}(0, R)$, such that*

$$|D^j(f - g_n)| \to 0 \ \text{as} \ n \to \infty, \quad \forall j \leq k,$$

uniformly on $\mathbb{B}(0, R)$.

Proof. Choose $\kappa_m \geq 1$ such that each nonempty closed subset $E \subset \mathbb{R}^d$ admits a $\{\kappa_m\}$ proxy distance. Let $\delta > 0$ be given, with $\delta < 1$, and let $\eta = 4\kappa^2\delta$ where $\kappa = \max\{\kappa_m : 0 \leq m \leq k\}$. Let $E_1, \ldots, E_n$ be the (η, E_{2R}) equivalence classes. Let $d_1, \ldots, d_n$ be $\{\kappa_m\}$ proxy distances for $E_1, \ldots, E_n$, respectively.

Let ϕ (depending on δ) be a C^∞ function from $\mathbb{R}$ into $[0, +\infty)$ such that

$$\phi(t) = 1 \text{ for } t \leq \kappa\delta,$$
$$\phi(t) = 0 \text{ for } t \geq 2\kappa\delta,$$
$$0 \leq \phi \leq 1 \ \text{ on } \mathbb{R},$$
$$|\phi^{(j)}| \leq c_j/\delta^j \ \text{ on } \mathbb{R}, \quad \forall j \geq 1,$$

where the c_j $(j = 1, 2, 3, \ldots)$ are independent of δ. Pick $p_j \in E_j$, for each j, and let

$$g_j(x) = \phi(d_j(x)) \cdot (f(x) - f(p_j)), \quad \forall x \in \mathbb{R}^d,$$

$$f_\delta = f - \sum_{j=1}^{n} g_j.$$

Then $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$. We claim that f_δ is locally-constant near E_R and that $|D^j(f - f_\delta)| \to 0$ as $\delta \downarrow 0$, $\forall j \leq k$, uniformly on $\mathbb{B}(0, R)$.

To see this, we set

$$U_j = \{x \in \mathbb{R}^d : d_j(x) < 2\kappa\delta\} \quad \text{and} \quad V_j = \{x \in \mathbb{R}^d : d_j(x) \in [\kappa\delta, 2\kappa\delta)\}.$$

Then U_j is an open neighbourhood of E_j,

$$\begin{aligned}
&\operatorname{dist}(x, E_j) < 2\kappa^2\delta = \eta/2, \quad \forall x \in U_j, \\
&U_j \cap U_i = \emptyset \quad \text{whenever } j \neq i, \\
&V_j \subset U_j, \\
&U_j \setminus V_j \quad \text{is an open neighbourhood of } E_j, \\
&\operatorname{spt} g_j \subset U_j, \\
&g_j = f - f(p_j) \quad \text{on } U_j \setminus V_j,
\end{aligned}$$

and it is clear from these properties that f_δ is locally constant near each E_j, and hence near E_R.

Furthermore, by Lemma 11,

$$|f - f(p_j)| \leq 2^{2d+1} M R^d \eta^{k+1} \quad \text{on } E_j,$$

where

$$M = \sup_{\mathbb{B}(0,2R)} \max_{j \leq k+d+1} |D^j f|.$$

Thus, since f is flat on E_j, we get

$$|f(x) - f(p_j)| \leq 2^{2d+1} M R^d \eta^{k+1} + M \operatorname{dist}(x, E_j)^{k+1} \leq c_0 M \delta^{k+1}$$

for $x \in U_j$, where c_0 depends only on d and k.

We also have

$$|D^i(f(x) - f(p_j)| = |D^i f(x)| \leq c_i M \delta^{k-i+1}$$

for $x \in U_j$, where c_j depends only on d, k and i.

One verifies inductively that

$$|D^i(\phi \circ d_j)(x)| \leq c'_i d_j(x)^{-i}, \quad \forall x \in \mathbb{R}^d, \quad \forall i \leq k,$$

where the c'_i depend only on d and i.

Thus we obtain the estimate

$$|D^i g_j| \leq c \cdot M \cdot \delta \text{ on } \mathbb{R}^d, \quad \forall i \leq k, \quad \forall j \in \{1, \ldots, n\},$$

where c depends only on d and k. Thus, since the supports of the g_j are disjoint,

$$|D^i(f - f_\delta)| \leq c \cdot M \cdot \delta \text{ on } \mathbb{R}^d, \quad \forall i \leq k.$$

This yields the result. □

References

[A] Allan, G., Kakiko, G., O'Farrell, A. G., and Watson, R. O., Finitely-generated algebras of smooth functions, in one dimension, J. Funct. Anal., to appear.

[N1] Nachbin, L., Sur les algèbres dense de fonctions différentiables sur une variété, C. R. Acad. Sci. Paris 228 (1949), 1549–1551.

[N2] Nachbin, L., A look at approximation theory, in: Approximation Theory and Functional Analysis (ed. by J. B. Prolla), 309–331, North Holland, 1979.

[T1] Tougeron, J. C., An extension of Whitney's spectral theorem, Publ. Math. I. H. E. S. 40 (1971), 139–148.

[T2] Tougeron, J. C., Ideaux de fonctions differentiables, Springer-Verlag, 1972.

[W] Whitney, H., On ideals of differentiable functions, Amer. J. Math. 70 (1948), 635–658.