THE 1-REDUCTION FOR REMOVABLE SINGULARITIES, AND THE NEGATIVE HÖLDER SPACES

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ABSTRACT

Let L be a pseudodifferential operator of (possibly non-integral) order m on $\mathbb{R}^d$, and let F be a topological vector space of distributions on $\mathbb{R}^d$. We say that a compact set $K \subset \mathbb{R}^d$ is *L–F-null* if, given an open set $U \subset \mathbb{R}^d$, and a distribution $f \in F$ such that Lf is C^∞ on $U \sim K$, it follows that Lf is C^∞ on U. We also say that K is a *set of removable singularities* for solutions of $Lf = g(g$ smooth) that belong to F. We describe the 1-reduction, a method for reducing problems about singularities for general elliptic operators to problems about supports. Applying the 1-reduction and a theorem of Dahlberg, we identify the L–T_β-null sets for all elliptic L and all real β such that $\beta \neq \text{order}L$. The scale of T_β is essentially an extension of the scale of Hölder–Zygmund spaces. These null-sets are the compact sets K with $M^{d+\beta-\text{order}L}(K) = 0$, where M^α denotes α-dimensional Hausdorff content. We indicate the further application of the 1-reduction to Sobolev and Besov spaces, and dual approximation problems.

1. Informal introduction

This paper is a fragment of a programme that has been running for the last century or so. You could even say that it has been running since Cauchy proved that isolated singularities are removable for bounded analytic functions. Let L be a pseudodifferential operator on $\mathbb{R}^d$, and let F be a topological vector space of distributions on $\mathbb{R}^d$. We say that a compact set $K \subset \mathbb{R}^d$ is *L–F-null* if, given an open set $U \subset \mathbb{R}^d$, and a distribution $f \in F$ such that Lf is C^∞ on $U \sim K$, it follows that Lf is C^∞ on U. We also say that K is a *set of removable singularities* for solutions of $Lf = g(g$ smooth) that belong to F. The programme is the more-or-less systematic assault on the question: describe the L–F-null sets as explicitly as possible, for each reasonable L and F. The fragment is the solution for all elliptic L and the case $F = \mathbf{Lip}\beta$ (for β non-integral and $\beta \neq \text{order}L$). Essentially, a nowhere-dense compact set $K \subset \mathbb{R}^d$ is L–$\mathbf{Lip}\beta$-null if and only if K has zero $(d+\beta-\text{order}L)$-dimensional Hausdorff measure. This actually works, not only for $0 < \beta < 1$, but also for $-\infty < \beta < +\infty$. That is, there is a reasonable definition of '$\mathbf{Lip}\beta$' for all real β, and with this definition, the thing works.

If we make the assumptions that (1) F contains the test functions and (2) the operator L is such that each C^∞ function g may be written locally as Lh, with $h \in C^\infty$, then (as is clear) K is L–F-null provided $Lf = 0$ on $U \sim K$ entails $Lf = 0$ on U. This assumption (2) is valid for all elliptic *differential* operators L [8, 17.1.1].

The method we shall use to establish the result is called the 1-reduction. It may be applied to a considerably more general class of function spaces F, but not to all. The crucial factor is the behaviour of the function space under singular integral operators.

Application of the 1-reduction to this problem reduces it to a problem previously solved by Dahlberg.

The 1-reduction may also be applied to problems other than removability, notably approximation problems and boundary smoothness problems.

Up to now, the state of public knowledge on this subject was as follows. The result is obviously trivial for $\beta > \text{order}L$, because in that case all nowhere-dense compacta are removable, and all sets in $\mathbb{R}^d$ have zero $(d+\beta-\text{order}L)$-dimensional Hausdorff measure. At the other extreme, we will see that the result is equally trivial for $\beta \leqslant \text{order}L - d$, because solutions of $Lf = 0$ belonging to '**Lip**β' may have isolated singularities, so that only the empty set is null. Apart from these trivial cases, the result was known for positive non-integral β and the two most important elliptic differential operators. For the Cauchy–Riemann operator $\bar{\partial}$ *on* $\mathbb{C} = \mathbb{R}^2$, Dolženko [5] proved the case $0 < \beta < 1$. For the Laplacian on $\mathbb{R}^d$, Carleson [3] proved the case $0 < \beta < 1$, and Verdera [17] the case $1 < \beta < 2$. More generally, Verdera proved the case of homogeneous constant-coefficient L and $(\text{order}L) - 1 < \beta < \text{order}L$. The case $L = \frac{d}{dx}$ could be described as folklore. The statement in that case is that there is a non-constant **Lip**β function, locally-constant on $\mathbb{R} \sim K$, if and only if K has positive β-dimensional measure. This is suitable as a homework problem for a real-variables course, and no doubt has so been used. As a specific example, there is a non-constant **Lip**$(\log_3 2)$ function, locally-constant off the usual Cantor set. The case $L = \frac{d^k}{dx^k}$ in $\mathbb{R}$ relates to the smoothness of 'splines'. One direction of the result, namely that null-sets of the appropriate Hausdorff measure are removable, works for arbitrary smooth differential operators, and systems of operators, and was proved by Harvey and Polking [6, theorem 4.5]. The other direction fails in this generality. For instance, it fails for $\bar{\partial}$ in $\mathbb{C}^2$, where there exist sets of Hausdorff dimension 4 that are $\bar{\partial}$–Lipα-null for all $\alpha > 0$.

The proof by 1-reduction bears little resemblance to the classical proofs of special cases.

It is pretty well known that there is a duality between questions of removable singularities and questions of approximation. Essentially, if the C^∞ functions are dense in a 'reasonable' space F, and $K \subset \mathbb{R}^d$ is compact and nowhere-dense, then the following are equivalent:

(1) the set $\{ f \in F : Lf = 0 \text{ near } K \}$ is dense in F on compacts;

(2) K is E–F^*-null, where E is a local inverse for L and F^* is the dual of F.

For instance, this was written out for general elliptic differential operators and the case of the $\mathbf{L}^p$–$\mathbf{L}^q$ duality by Serrin. It has been extensively exploited in the $\mathbf{L}^p$ case by Havin, Hedberg, Bagby, Polking, and others, mostly for special differential operators. Oddly enough, this insight has served to obscure more than to illuminate the essentials of removable singularity theory. The essence of our method in this paper is the use of a different equivalence, which can be expressed (loosely speaking) thus:

$$K \text{ is } L-F\text{-null} \Leftrightarrow K \text{ is } 1-LF\text{-null}.$$

Here 1 stands for the (elliptic) operator

$$1 : f \mapsto f.$$

To say that $1f = 0$ on an open set U is simply to say that the support of f is disjoint from U. By LF we just mean

$$\{Lf : f \in F\}.$$

Thus, all questions about removable singularities are reduced to questions about the 1-operator. For 'reasonable' F and elliptic L the space LF depends only on (F and on) the order of L (up to local equivalence—see below). In the present case, $\mathbf{Lip}\beta$ is 'reasonable' for $0 < \beta < 1$, and for $m \in \mathbb{Z}$, the common value of $L\mathbf{Lip}\beta$, over all L of order m, is basically what we call $\mathbf{Lip}(\beta - m)$; however, we will give a much more explicit description of it.

It turns out that the problem of describing the 1–$\mathbf{Lip}\beta$-null sets in $\mathbb{R}^d$ is non-trivial precisely when β is in the range $-d < \beta \leqslant 0$. Thus the $\mathbf{Lip}\beta$ for negative β play a crucial role. Take, for example, the bi-Laplacian, $L = \Delta^2$, important in elasticity theory. The interesting range of $\mathbf{Lip}\beta$s is given by $4 - d < \beta \leqslant 4$. For $\beta < 4$, a compact set K is $\Delta^2 - \mathbf{Lip}\beta$-null if and only if $M^{d+\beta-4}(K) = 0$. In case the dimension $d = 2$, this tells you how smooth an elastic plate can be which is clamped along a set K, in terms of the Hausdorff dimension of K. Note that for $d > 4$, there is a non-trivial problem for the negative Hölder classes. The same thing happens for the Laplacian when the dimension exceeds 2. For instance, in dimension 3, one-half-dimensional Hausdorff measure tells you about removable singularities for $\mathbf{Lip}(-\frac{1}{2})$ harmonic functions. From the physical point of view, it tells you about the electrical field of a charge distributed on a fractal. By duality, it tells you about approximation by Newtonian potentials in pre-duals of $\mathbf{Lip}(-\frac{1}{2})$. One such pre-dual is, in fact, a certain Besov space, known as $\mathbf{B}^{\frac{1}{2}}_{1,1}$ (locally-equivalent to the enveloping Banach space of the Newtonian potentials of elements of the pseudonormed Hardy space $H^{\frac{1}{2}}(\mathbb{R}^d)$).

Triebel [15] showed that for $0 < \alpha < 1$, the space $\mathbf{L}^\infty \cap \mathbf{Lip}\alpha$ coincides with the Besov space $\mathbf{B}^\alpha_{\infty,\infty}$ (defined in terms of properties of a Paley–Littlewood-style decomposition). The scale $\mathbf{B}^\beta_{\infty,\infty}(-\infty < \beta < +\infty)$ thus provides a natural version of '**Lip**β' for each real β, and the removability result holds for it. But adopting this definition does not materially reduce the amount of work we have to do to establish the result, and it adds baggage, so we prefer to use a more elementary version T_β of **Lip**$\beta(-\infty<\beta<+\infty)$. The ideas in this paper are fundamentally simple, so we are anxious to avoid obscuring them by making the development depend on the rather elaborate theory of the Besov spaces. In the final two sections of the paper, we assume familiarity with the Besov spaces, prove the duality result and indicate some further applications of the 1-reduction.

2. Formal introduction

We consider linear differential operators on $\mathbb{R}^d$ of the form

$$Lf=\sum_{|i|\leqslant m} a_i(x)\partial^i f.$$

Here $i=(i_1,\ldots,i_d)$ denotes a multi-index and $|i|=\sum_j i_j$ denotes its rank. We consider only the case of C^∞ coefficients a_i, although this is not a crucial assumption. We call the operator L *elliptic* of order m if, for each $x\in\mathbb{R}^d$, the homogeneous polynomial

$$\xi\to p_m(x,\xi)=\sum_{|i|=m} a_i(x)\iota^{|i|}\xi^i$$

has no real zero $\xi\in\mathbb{R}^d$, apart from $\xi=0$. This polynomial-valued function $p_m(x,\xi)$ is called the principal symbol of the operator L.

We understand the term *pseudodifferential operator* on $\mathbb{R}^d$ in the sense used in [8, chap. XVIII]. A pseudodifferential operator S of order m takes the form

$$f\mapsto(2\pi)^{-d}\cdot\mathfrak{F}^{-1}(\xi\mapsto p(x,\xi)\cdot(\mathfrak{F}f(\xi))$$

where $\mathfrak{F}$ denotes the Fourier transform and $p(x,\xi)$ is a smooth *symbol*, which satisfies certain estimates. These are that for each of the multi-indices j and k, and each compact set $K\subset\mathbb{R}^d$,

$$\left|\frac{\partial^{|j|}}{\partial x^j}\frac{\partial^{|k|}}{\partial\xi^k}p(x,\xi)\right|\leqslant\kappa(K,j,k)(1+|\xi|)^{m-|k|}$$

whenever $x\in K$ and $\xi\in\mathbb{R}^d$. In particular, such operators S are smooth, in the sense that they map C^∞ functions having compact support to C^∞ functions. Also, S corresponds to a Schwartz kernel that is smooth away from the diagonal [8, p.69]. Thus if $f\in\mathscr{E}'$, then Sf is smooth away from the support of f.

Note that pseudodifferential operators may be of non-integral order. An operator that is of *every* real order is said to be of order $-\infty$.

We call a pseudodifferential operator of order m with symbol $p(x,\xi)$

elliptic of order m if for each compact $K \subset \mathbb{R}^d$ there exist $\kappa > 0$ and $R > 0$ such that

$$|p(x, \xi)| > \kappa|\xi|^m \text{ whenever } x \in K,\ |\xi| > R.$$

For $0 < \alpha \leqslant 1$, **Lip**α denotes the space of those $f : \mathbb{R}^d \to \mathbb{C}$ for which there exists $\kappa > 0$ such that

$$|f(x) - f(y)| \leqslant \kappa|x-y|^\alpha,\ \forall x,\ y \in \mathbb{R}^d.$$

Lipα becomes a Banach space when given the norm

$$\| f \|_{\mathbf{Lip}\alpha} \stackrel{\text{def}}{=} | f(0)| + \text{least}\,\kappa.$$

For $k \in \mathbb{N}$ and $0 < \alpha \leqslant 1$, **Lip**$(k+\alpha)$ denotes the space of those $f \in \mathbf{Lip}\alpha$ such that all partial derivatives $\partial^i f$ of order $|i| \leqslant k$ also belong to **Lip**α.

The space **Lip**β is 'locally equivalent' (see below) to the space $\mathbf{L}^\infty \cap \mathbf{Lip}\beta$ of bounded elements of **Lip**β, and also to the space $\mathbf{Lip}\beta_{\text{cs}}$ of compactly-supported elements of **Lip**β. As a result, for each differential operator L, the L–**Lip**β-null sets are precisely the same as the L–$(\mathbf{L}^\infty \cap \mathbf{Lip}\beta)$-null sets, and as the L–$\mathbf{Lip}\beta_{\text{cs}}$-null sets. This also works for pseudodifferential operators, S, but is slightly less obvious. It depends upon the fact that Sf is smooth away from the support of f.

Taibleson [13] proved that for $0 < \beta \notin \mathbb{Z}$, and any fixed integer $k > \beta$, $\mathbf{L}^\infty \cap \mathbf{Lip}\beta$ may be described as the space of those $f \in \mathbf{L}^\infty$ such that there exists a constant $\kappa > 0$ (depending on f), with

$$\left|\frac{\partial^k}{dt^k} P_t * f\right| \leqslant \kappa t^{\beta-k},\ \forall t > 0, \tag{1}$$

where P_t denotes the Poisson kernel. There is a detailed account in [11]. Practically the same proof (cf. section 3 below) shows that $f \in \mathbf{Lip}\beta_{\text{cs}}$ if and only if $f \in \mathcal{E}'$ and there exists $\kappa > 0$ such that inequality (1) holds for each (or any one) integer $k > \beta$. For $\beta \in \mathbb{R}$, we use T_β to denote the space of those $f \in \mathcal{E}'$ such that there exists $\kappa > 0$ satisfying (1) for each (or any one) non-negative integer $k > \beta$. Thus $T_\beta = \mathbf{Lip}\beta_{\text{cs}}$ for $\beta > 0$, and for non-positive β, T_β provides a natural way to extend the scale of $\mathbf{Lip}\beta_{\text{cs}}$ spaces. For $\beta < 0$, distributions $F \in T_\beta$ have compact support and satisfy the estimate

$$|\mathbf{P}_t * f| \leqslant \kappa t^\beta,\ \forall t > 0, \tag{2}$$

for some constant $\kappa \geqslant 0$.

The spaces T_β for negative β are the 'negative Hölder spaces' of the title. We are going to present a general removable-singularities result for these spaces.

Those experienced in function space theory will here be reminded of the controlled-mean-value spaces introduced by Morrey. Morrey's spaces were, however, spaces of locally-integrable functions, and were not complete in the natural topology. The spaces T_β are spaces of more general distributions, and are complete. Also, the index β is unrestricted here, whereas with the Morrey spaces it was only sensible to consider β down to $-d$. See also [16, 183–4], where a similar limitation is observed.

A few examples of negative order Hölder functions are in order. On $\mathbb{R}$, the Dirac delta functional

$$f \mapsto f(0)$$

belongs to T_{-1}, and its derivative

$$f \mapsto -\frac{df}{dx}(0)$$

belongs to T_{-2}. The principal value functional

$$f \mapsto P.V. \int \frac{f(x)}{x}\, dx$$

(the 'function' $1/x$) also belongs to T_{-1}. If a function on $\mathbb{R}^d$ has compact support and is bounded, then it belongs to every T_β with $\beta < 0$. On $\mathbb{R}$ a function in $\mathbf{L}^1_{cs}$ belongs to T_β whenever $\beta \leqslant -1$. On $\mathbb{R}^2$, this is not true, but it is true that all $\mathbf{L}^2$ functions with compact support belong to T_{-1}. On $\mathbb{R}$, the usual measure supported on the Cantor set ($\log_3 2$-dimensional Hausdorff measure) belongs to T_α where $\alpha = \log_3 \frac{2}{3}$. A function on $\mathbb{R}^2$ representing an ordinary crack, i.e. continuous except for a jump discontinuity across a smooth curve, would belong locally to T_{-1}.

The results in this paper will apply to the spaces T_β in many cases of integral β, but these are not classical Lipschitz spaces. For integral β, the spaces T_β are closely related to the Zygmund class. For $\beta = k$, a positive integer, T_β is locally-equivalent to the space of those functions having compact support whose derivatives of order $k-1$ lie in the Zygmund class. The space T_0 is rather close to, but not identical with, the space **BMO**. (It has been described as a Bloch space, but that term is often reserved for the case $d = 1$ and situation where the Poisson transform $P_t * f$ is *analytic* in the upper half-plane.) It contains some distributions that are not representable by integrable functions. Thus, we have nothing to say here about the L–**Lip**k-null sets ($k = 1,2,3, \ldots$) or the L–**BMO**-null sets.

For properties of Hausdorff content M^β, we refer to [3]. As a matter of convenience, we define M^β for $\beta < 0$ to be M^0. Observe that $M^0(E)$ equals 0, 1, or $+\infty$ according as E is empty, non-empty and bounded, or unbounded.

We now state the main result, the 1-reduction, in this context.

Theorem 1. *Let K be a compact subset of $\mathbb{R}^d$, and let $\beta \in \mathbb{R}$. Let S be a* *pseudodifferential operator of order m on* $\mathbb{R}^d$. *Then K is* $S-T_\beta$*-null if K is* $1-T_{\beta-m}$*-null. If, in addition, S is elliptic of order m, then, conversely, K is* $1-T_{\beta-m}$*-null if K is* $S-T_\beta$*-null.*

The characterisation of the $1-T_\beta$-null sets is due to Dahlberg [4], and is as follows.

Dahlberg's Theorem. *Let K be a compact subset of* $\mathbb{R}^d$, *and let* $0 \neq \beta \in \mathbb{R}$. *Then the following are equivalent*:

(1) *K is* $1-T_\beta$*-null, i.e.* 0 *is the only distribution belonging to* T_β *with support in K*;

(2) $M^{d+\beta}(K)=0$.

Dahlberg, in fact, proved a more general result about subharmonic functions in Lipschitz domains. The restriction that $\beta \neq 0$ is necessary. If $\beta = 0$, then it is known that (1) implies (2), but not conversely.

Combining Dahlberg's Theorem and the 1-reduction, we obtain the following corollary.

Corollary 2. *Let K be a compact subset of* $\mathbb{R}^d$, *and let* $\beta \in \mathbb{R}$. *Let S be an elliptic pseudodifferential operator of order m on* $\mathbb{R}^d$. *Suppose* $\beta \neq m$. *Then K is* $S-T_\beta$*-null if and only if* $M^{d+\beta-m}(K)=0$.

The result for *differential* operators is worth stating separately.

Corollary 3. *Let K be a compact subset of* $\mathbb{R}^d$, *and let* $\beta \in \mathbb{R}$. *Let L be an elliptic differential operator of order m on* $\mathbb{R}^d$ *having* C^∞ *coefficients. Suppose* $\beta \neq m$. *Then K is* $L-T_\beta$*-null if and only if* $M^{d+\beta-m}(K)=0$.

We remark that the result is essentially local, and so transfers automatically from $\mathbb{R}^d$ to arbitrary C^∞ manifolds.

Corollary 2 has some interest when applied to a parametrix for an elliptic differential operator. By duality it then yields an approximation theorem in the Besov space $B^s_{1,1}$. We state the more general version for pseudodifferential operators. This applies as well, for instance, to approximation by some potentials and by Cauchy transforms. There is a possible global obstacle to the approximation, so we state the result in terms of the space $B^s_{1,1\mathrm{loc}}$, of functions that belong locally to $B^s_{1,1}$. This is a Fréchet space, when topologised in the natural way. We also impose a technical restriction.

Corollary 4. *Let* $s \in \mathbb{R}$, *and let S be an elliptic pseudodifferential operator of order m, admitting pseudodifferential operators Q and R of order* $-m$ *such that* $QS = SR = 1$ *on the compactly-supported distributions. Suppose* $s+m$ *is non-zero. Then the following are equivalent*:

(1) *each* $f \in B^s_{1,1\mathrm{loc}}$ *is the limit of a sequence of functions* $\phi_n \in \mathscr{D}$ *which*

satisfy $S\phi_n = 0$ *near* K;

(2) $M^{d+m-s}(K) = 0$.

The restriction that there exist parametrices Q and R of the type specified can be relaxed, but as it stands it covers many interesting examples. It is satisfied for all translation-invariant operators, in particular.

Our approach to the proof of Corollary 2 differs from the classical proofs of the known special cases, and thus provides a new way to view those cases. Here are some further examples, beginning with an example on Dahlberg's Theorem.

Example 1. The 1 operator on $\mathbb{R}$.

We are talking here about the harmonic extension to the upper half-plane of distributions on $\mathbb{R}$. The main consequence of Theorem 1 is that for $\beta > 0$ the compact $K \subset \mathbb{R}$ supports f with

$$|P_t * f| \leqslant \frac{K}{t^{\beta}}, \ \forall t > 0$$

if and only if $M^{1-\beta}(K) > 0$. The interesting range of β is $0 < \beta < 1$. For $\beta \geqslant 1$, each non-empty K supports such distributions.

Example 2. The $\bar{\partial}$ operator on $\mathbb{C}$. For $\beta \neq 1$, the compact K is $\bar{\partial} - T_\beta$-null if and only if $M^{1+\beta}(K) = 0$.

For $0 < \beta < 1$, this is Dolzhenko's Theorem. The result is new for $-1 < \beta \leqslant 0$. It explains the real significance for analytic functions of the condition $M^{\alpha}(K) = 0$, when $0 < \alpha < 1$. Carleson [2] found a connection between this condition and a Hölder class of *multiple-valued* analytic functions, related to analytic differentials. The present result shows that the removability theorems for forms and functions differ by an order of smoothness.

We mention in passing the rather deeper result of Nguyen [10] that the $\bar{\partial}$–**Lip**1-null sets are also the sets of area zero. Nguyen has also shown that the statement as given fails for $\beta = 1$: there do exist sets of area zero which are not $\bar{\partial} - T_1$-null.

One direction of the case $\beta = 0$ is implied by Kaufmann's theorem that $M^1(K) = 0$ characterises the $\bar{\partial}$–**BMO**-null sets. The result shows that the $\bar{\partial}$–**BMO**-null sets are the same as the (potentially-scarcer) $\bar{\partial} - T_0$-null sets.

Example 3. The Laplacian. For $\beta \neq 2$, the compact $K \in \mathbb{R}^d$ is $\Delta - T_\beta$-null if and only if

$$M^{d+\beta-2}(K) = 0.$$

The result yields new information whenever $d \geqslant 3$. It concerns the growth in the half-space $\mathbb{R}^{d+1}_+$ of the Poisson transform of distributions harmonic on $\mathbb{R}^d \sim K$. It also yields new information in the case $\beta = 1$, $d \geqslant 3$.

The case $\beta = 1$, $d = 2$ (the Zygmund class in the plane) was previously known to Verdera, and was communicated privately to the author.

Example 4. $\bar{\partial}^2$ on $\mathbb{C}$.

Solutions of $\bar{\partial}^2 f = 0$ take the form $g(z) + h(z)\cdot\bar{z}$, with g and h analytic. These have been used as 'complex potentials' to study elastic plates, in essentially the same way as analytic functions are used in connection with ideal fluids. The non-null-sets are then possible singularities in solutions to the elasticity equation, i.e. loci along which the plate is 'clamped' or 'cracked'. Corollary 3 shows that null-sets for $\bar{\partial}^2$ and $T_\beta (\beta \neq 2)$ are precisely the same as for the Laplacian.

Apart from the cases $\beta = 1$ of Example 3 and $0 < \beta \leqslant 1$ of Example 4, the examples so far involve no new information about classical functions, as opposed to distributions. More new facts about genuine functions appear when we consider operators of order at least 3. We have already mentioned the bi-Laplacian. Here is one more.

Example 5. $\bar{\partial}^3$ on $\mathbb{C}$.

This concerns removable singularities for functions locally representable as $f(z) + \bar{z}g(z) + \bar{z}^2 h(z)$ with f, g and h analytic. The result is that for $\beta \neq 3$, K is removable for such functions of class T_β if and only if $M^{\beta - 2}(K) = 0$. The interesting range is $1 < \beta < 3$. The range $2 < \beta < 3$ is covered by Verdera's theorem, and the range $1 < \beta \leqslant 2$ is new.

Example 6. The Cauchy transform.

As an example of a non-differential operator, consider the Cauchy transform, $\mathfrak{C}$ on $\mathbb{C}$. This is a parametrix for $\bar{\partial}$ and is elliptic of order -1. Strictly speaking, it is not covered by the discussion, because its symbol has a singularity at the origin. In general, if $p(x, \xi)$ is the symbol of an elliptic pseudodifferential operator, then $1/p(x, \xi)$ is not usually smooth, and must be modified (e.g. by replacing it by $\psi(\xi)/p(x, \xi)$, where $\psi \in \mathscr{E}$, $\psi = 0$ near 0, and $\psi = 1$ off a ball) in order to obtain a symbol for a parametrix. If we do this to $\mathfrak{C}$, we get an operator which differs from $\mathfrak{C}$ by a smoothing operator, and then we can apply Corollary 2. The result is that for $\beta + 1 \neq 0$, the $\mathfrak{C} - T_\beta$-null compact sets are those with $M^{3+\beta}(K) > 0$. The interesting range is $-3 < \beta < -1$, in other words we are talking about rather bad distributions. But that means that the dual approximation result is about rather nice functions. It states:

For $s \neq 1$, the smooth functions holomorphic near a compact $K \in \mathbb{C}$ are dense in $B^s_{1,1}$ if and only if $M^{3-s}(K) = 0$.

Focussing on the positive s, we see that the approximation is always possible when $0 < s < 1$, and never possible (unless $K = \emptyset$) when $s \geqslant 3$, and that for $1 < s < 3$ it depends on the $(3-s)$-dimensional Hausdorff content. This provides yet another answer to the question (cf. Example 2): what does M^β have to do with holomorphic functions? The same kind of analysis applied to the Newtonian potential yields an approximation theorem for harmonic functions:

For $s \neq 2$, the smooth functions harmonic near a compact $K \in \mathbb{R}^d$ are dense in $B^s_{1,1}$ if and only if $M^{d+2-s}(K) = 0$.

We remark that the spaces $B^s_{1,1}$ are related to derivatives and potentials of Hardy spaces.

Also, the results presented in this paper have counterparts for the 'little lip' spaces. The Hausdorff content has to be replaced by the lower Hausdorff content. The modifications are routine.

3. Function space properties

In this section we assemble the facts we shall need about function spaces and the mapping properties of pseudodifferential operators.

We denote the space of test functions on $\mathbb{R}^d$ by $\mathscr{D}$, and the space of distributions by $\mathscr{D}'$. We regard $\mathscr{D}$ as a subspace of $\mathscr{D}'$, in the usual way (i.e. by identifying $f \in \mathscr{D}$ with the distribution

$$\phi \mapsto \int_{\mathbb{R}^d} \phi f dx).$$

The product of the test function ϕ and the distribution f is the distribution ϕf, given by

$$\langle \psi, \phi f \rangle = \langle \psi\phi, f \rangle, \ \forall \psi \in \mathscr{D}.$$

The complex conjugate $\bar{f}$ of a distribution f is defined by

$$\langle \psi, \bar{f} \rangle = \overline{\langle \bar{\psi}, f \rangle}, \ \forall \psi \in \mathscr{D}.$$

We denote the group of invertible affine transformations of $\mathbb{R}^d$ by **Aff**. For $T \in \mathbf{Aff}$ and $f \in \mathscr{D}'$, we define the composition $f \circ T$ by

$$\langle \psi, f \circ T \rangle = (\det T)^{-1} \cdot \langle \psi \circ T^{-1}, f \rangle, \ \forall \psi \in \mathscr{D}.$$

A *symmetric concrete space* (SCS) is a complete locally-convex topological vector space F such that

(1) $\mathscr{D} \hookrightarrow F \hookrightarrow \mathscr{D}'$;

(2) F is a topological $\mathscr{D}$-module, under the above action;

(3) $f \mapsto \bar{f}$ is a TVS automorphism of F;

(4) for each $T \in \mathbf{Aff}$ the map $c_T : f \mapsto f \circ T$ is a TVS automorphism of F, and the map $T \mapsto c_T$ sends compact subsets of **Aff** to equicontinuous sets of automorphisms.

This is what we mean by a 'reasonable' space.

When regarded as spaces of distributions, **Lip**β ($\beta > 0$) and T_β (β real) are SCS, as is readily seen.

Given an SCS, F, we construct the spaces

$$F_{cs} = \mathscr{D} \cdot F$$

$$F_{loc} = \mathscr{E} \cdot F.$$

With the natural topologies, these are also SCS.

We say that two SCS, F_1 and F_2, are *locally-equivalent* if $F_{1loc} = F_{2loc}$. We use the notation $F_1 \overset{loc}{=} F_2$. For such spaces, the L–F_1-null sets coincide with the $L-F_2$-null sets, for each differential operator L. This is evident.

Lipβ is, of course, locally-equivalent to **Lip**β_{cs}, and Taibleson's theorem implies that **Lip**$\beta_{cs} = T_\beta$ when $0 < \beta \notin \mathbb{Z}$. We need the fact that T_β may be described in several superficially different ways.

Lemma 5. *Let β be real, let k, l and m be integers greater than β, and let f be a distribution on $\mathbb{R}^d$ having compact support. Then the following are equivalent.*

(1) *There exists $\kappa > 0$ such that*

$$\left|\frac{\partial^k}{\partial t^k} P_t * f(x, t)\right| \leqslant \kappa \cdot t^{\beta-k}$$

whenever $x \in \mathbb{R}^d$ and $t > 0$.

(2) *There exists $\kappa > 0$ such that*

$$\sum_{j=1}^{d} \left|\frac{\partial^l}{\partial x_j^l} P_t * f(x, t)\right| \leqslant \kappa \cdot t^{\beta-l}$$

whenever $x \in \mathbb{R}^d$ and $t > 0$.

(3) *There exists $\kappa > 0$ such that*

$$\sum_{|i|=m} \left|\frac{\partial^m}{\partial x^i} P_t * f(x, t)\right| \leqslant \kappa \cdot t^{\beta-m}$$

whenever $x \in \mathbb{R}^d$ and $t > 0$.

PROOF. This is proved in almost precisely the same way as the corresponding facts in [11, pp 142 and following]. Stein is working under the hypothesis $f \in \mathbf{L}^\infty$, but the hypothesis that f has compact support does just as well. For instance, on p. 144 we still get

$$\frac{\partial u}{\partial x_j} \to 0 \text{ as } y \uparrow \infty,$$

because

$$\frac{\partial P_y}{\partial x_j} \to 0 \in \mathscr{E} \text{ as } y \uparrow \infty,$$

uniformly over all translates, because

$$\begin{aligned}\frac{\partial u}{\partial x_j}(x) &= \left(\frac{\partial P_y}{\partial x_j} * f\right)(x)\\ &= \left\langle \frac{\partial P_y}{\partial x_j}(x - \cdot), f \right\rangle\\ &\to 0 \text{ uniformly in } x. \blacksquare\end{aligned}$$

The natural SCS topology on T_β represents it as the strict inductive limit of the Banach spaces $\{f \in \mathbf{Lip}\beta : \mathrm{spt} f \subset K\}$, where K runs over all compact subsets of $\mathbb{R}^d$. The fact that it is not actually a Banach space will not cause us any difficulty.

Differential operators of order m map T_β continuously to $T_{\beta-m}$, but that is too much to ask of pseudodifferential operators.

For an operator A and an SCS F, we say that F is *weakly-locally A-invariant* if A maps F_{cs} continuously into F_{loc}.

Theorem 6. *If S is a pseudodifferential operator of order $m \in \mathbb{R}$, and $\beta \in \mathbb{R}$, then S maps T_β continuously into $T_{\beta - m\,\mathrm{loc}}$.*

Remarks. This theorem seems to be widely known, but I have not found it written down in this complete form. The manner in which pseudodifferential operators are defined is designed to ensure that an operator of order m is roughening of order at most m (smoothing of order $-m$) on the scale of $\mathbf{L}^2$ Sobolev spaces:

$$H^s = (\mathfrak{F})^{-1}[(1 + |\xi|^2)^{-s/2} \cdot \mathfrak{F}\mathbf{L}^2],$$

i.e. maps $H^s \to H^{s-m}$. This property is enough for many applications, and most people prefer to use it than to go to the trouble of establishing Schauder-style bounds.

Proof. For differential operators, the result is easy.

For the case where S is translation-invariant and $\beta > 0$ the result is very old. The case of non-integral $\beta > 0$ is more or less due to Frostman, and Zygmund [18] essentially did the integral case. To quote a specific proof, [12] contains a short clever proof of the analogue on the torus of the case $m = 0$, $0 < \beta \leqslant 1$. On the torus, the T_{cs} and T_{loc} coincide. In Euclidean space, the statement becomes false without the $_{\mathrm{cs}}$ and $_{\mathrm{loc}}$. But with this modification, Taibleson's proof carries over word for word. We will see below how to transfer the result from the range $\beta \in (0, 1]$ and $m = 0$ to all real β and m. Thus, for the case which is, after all, the most interesting, the result is quite accessible.

From the lore on the translation-invariant operators, we extract and record the case of the Bessel transforms. These are the operators J_m with symbols $(1+|\xi|^2)^{-m/2}$.

Lemma 7. *If $m\in\mathbb{R}$ and $\beta\in\mathbb{R}$, then J_m maps T_β into $T_{\beta+m\mathrm{loc}}$.*

PROOF. This may be seen by imitating the argument in [11, 149–50], employing, as with Lemma 5, the assumption of compact support in place of boundedness. ■

For the case where S is a parametrix for a *properly* elliptic differential operator of order $-m$, and $\beta>0$, the theorem is implied by a theorem in Triebel [16, (4.3.4) pp 235–6] (the apotheosis of the Schauder estimates), in view of the fact that $B^\beta_{\infty,\infty}$ is locally-equivalent to T_β. (In a *properly* elliptic operator the principal symbol $\xi\mapsto p(x,\xi)$ has *distinct* roots off the real space for each $x\in\mathbb{R}^d$.)

The general case with order$S=0$ of Theorem 6 is covered by the results in [14]. This proof uses Triebel's elaborate and powerful machine, and so lies quite deep. The case order$S=0$, $0<\beta<1$ is also covered by a theorem proved in Lemarie's thesis [9, II.2, thm A(b), p. 34], in view of the facts that (1) his spaces Λ^α are locally equivalent to $T_\alpha(0<\alpha<1)$ and (2) pseudodifferential operators of order 0 are examples of what he calls SIOs of order 0 and class 1. His proof is not all that short, but it is completely elementary. We state this case as a lemma.

Lemma 8. *If S is a pseudodifferential operator of order 0, and $0<\beta<1$, then T_β is weakly-locally S-invariant.*

Now we can complete the proof of Theorem 6. Let S be a pseudodifferential operator of any order $m\in\mathbb{R}$ and let β be any real number. Write $m=\gamma+\delta$, with $0<\beta-\gamma<1$. Let ϕ_1, ϕ_2, ϕ_3, ϕ_4, $\in\mathscr{D}$ be any test functions. Then $J_\delta\phi_2S\phi_3J_\gamma$ is a pseudodifferential operator of order 0, hence maps $T_{\beta-\gamma}$ into $T_{\beta-\gamma\mathrm{loc}}$. Applying Lemma 7, $A=J_{-\delta}\phi_1J_\delta\phi_2S\phi_3J_\gamma\phi_4J_{-\gamma}$ maps T_β into $T_{\beta-m\mathrm{loc}}$.

Given a closed ball B, consider ϕ_i that equal 1 near B. For $g\in\mathscr{E}'$, the differences

$$J_{-\delta}\phi_1J_\delta g-g$$

and

$$J_\gamma\phi_4J_{-\gamma}g-g$$

are smooth near B. Thus $Ag-Sg$ is C^∞ near B, so that S maps each $f\in T_\beta$ to a function that, near B, differs from something in $T_{\beta-m}$ by a smooth function. This is enough. ■

4. Proof of Theorem 1

We begin by recording a fundamental fact about the existence of parametrices (cf. [8, p.72]).

Lemma 9. *Let S be an elliptic pseudodifferential operator of order m. Then there exists a pseudodifferential operator Q of order $-m$ such that $SQ-1$ and $QS-1$ are pseudodifferential operators of order $-\infty$.* ■

Now we can finish the proof of Theorem 1.

(1) Let K be $1-T_{\beta-m}$-null.

Let $U\subset\mathbb{R}^d$ be open, $f\in T_\beta$, $g\in\mathscr{E}$ and $Sf=g$ on $U\sim K$.

Then, by Theorem 6, $Sf\in T_{\beta-m\text{loc}}$. Thus $Sf=g$ on U. That proves one direction.

(2) Assume S is elliptic. Let K be $S-T_\beta$-null.

Let $U\subset\mathbb{R}^d$ be open, $f\in T_{\beta-m}$, and $f=0$ on $U\sim K$. We wish to show that $f=0$ on U. We may assume that U is a ball.

By Lemma 9, there exists a pseudodifferential operator Q of order $-m$ such that $SQf=f+h$, where $h\in\mathscr{E}$. By Theorem 6, $Qf\in T_\beta$. Pick $\phi\in\mathscr{D}$ with $\phi=1$ on a neighbourhood of clos(U). Then

$$S(\phi\cdot Q(f))=f+h+S((1-\phi)Q(f))=f+k,\ \text{say}.$$

Since k is C^∞ on U, and T_β is a $\mathscr{D}$-module, we conclude that

$$S(\phi\cdot Q(f))=k \text{ on } U,$$

hence $f=0$ on U, as required. ■

5. Duality

Here we discuss the proof of Corollary 4, assuming familiarity with the Besov spaces. We use the notation of [16].

First, by a result of Triebel [15], $B^s_{\infty,\infty}$ is locally equivalent to T_s for $s>0$. Since $B^s_{\infty,\infty}$ is invariant under the classical Calderon–Zygmund SIO, this equivalence extends to all s, in view of the 'fundamental theorem of calculus' for function spaces, explained below. Thus we may replace T_β by $B^\beta_{\infty,\infty}$ in the statement of Corollary 2.

Given an SCS, F, we define new spaces

$$DF=\left\{\lambda+\frac{\partial f_1}{\partial x_1}+\cdots+\frac{\partial f_d}{\partial x_d}:f_1,\ldots,f_d\in F,\ \lambda\in\mathbb{C}\right\}$$

and

$$\int F=\left\{f\in\mathscr{D}':\frac{\partial f_1}{\partial x_1},\ldots,\frac{\partial f_d}{\partial x_d}\in F\right\}.$$

These become SCS when topologised in the obvious ways.

Lemma 10. *$DT_\beta \overset{\text{loc}}{=} T_{\beta-1}$ and $\int T_\beta \overset{\text{loc}}{=} T_{\beta+1}$ whenever $\beta \in \mathbb{R}$.*

PROOF. Apply the fact that condition (2) in Lemma 5 is equivalent for any two different $l > \beta$. ∎

Lemma 11. *If an SCS F is weakly-locally invariant under all pseudodifferential operators of order 0, then so are $\int F$ and DF.*

PROOF. Fix a pseudodifferential operator S of order 0. For $1 \leqslant j \leqslant d$, the commutator $[S, \partial^j]$ is a pseudodifferential operator of order 0, and hence maps F_{cs} to F_{loc}. This is enough. ∎

Lemma 12 (fundamental theorem of calculus). *Let F be a symmetric concrete space that is weakly-locally invariant under the classical Calderon–Zygmund singular integral convolution operators. Then*

$$\int DF \overset{\text{loc}}{=} F \overset{\text{loc}}{=} D \int F.$$

PROOF. For general SCS, F, we have

$$F \subset \int DF \text{ and } D \int F \subset F.$$

To see that $F \overset{\text{loc}}{\hookrightarrow} D\int F$, fix $f \in F_{cs}$. Let $\mathfrak{B}$ be the Newtonian potential, inverting the Laplacian. Then all second derivatives of $\mathfrak{B}f$ belong to F_{loc}, since they are Calderon–Zygmund singular integral operators applied to f. Thus

$$\frac{\partial}{\partial x_j} \mathfrak{B}f \in \int F \quad (j = 1, \ldots, d),$$

from which it follows that $f = \Delta \mathfrak{B} f$ belongs to $D\int F$.

To see that $\int DF \overset{\text{loc}}{\hookrightarrow} F$, fix $f \in (\int DF)_{cs}$. Fix $\phi \in \mathscr{D}$, equal to 1 on a neighbourhood of the support of f. There exist $g_{j,k} \in F$ such that

$$\frac{\partial f}{\partial x_j} = \sum_{k=1}^{d} \frac{\partial g_{j,k}}{\partial x_j}.$$

It follows easily that

$$f = \mathfrak{B}\Delta f = \sum_{j,k} \mathfrak{B} \frac{\partial^2}{\partial x_j \partial x_k} (\phi g_{j,k}),$$

which belongs to F_{loc}, since

$$\mathfrak{B}\,\frac{\partial^2}{\partial x_j \partial x_k}$$

is a Calderon–Zygmund SIO. ■

Continuing with the proof of Corollary 4, we now let Q and R be pseudodifferential operators of order $-m$ such that $QS = SR = 1$ when restricted to compactly-supported distributions.

The dual of $B^s_{1,1}$ is $B^{-s}_{\infty,\infty}$ [16, 178]. It follows that the dual of $B^s_{1,1\mathrm{loc}}$ is $B^{-s}_{\infty,\infty\mathrm{cs}}$. Condition (1) of Corollary 4 is thus equivalent to the statement: if $g \in B^{-s}_{\infty,\infty\mathrm{cs}}$ and if

$$\langle \phi, g\rangle = 0 \text{ whenever } \phi \in \mathscr{D} \text{ and } S\phi = 0 \text{ near } K, \tag{3}$$

then $g = 0$. Now condition (3) is equivalent to:

$$\langle S\phi, Q^t g\rangle = 0 \text{ whenever } \phi \in \mathscr{D} \text{ and } S\phi = 0 \text{ near } K. \tag{4}$$

Using R to see one direction, we find that (4) is equivalent to

$$\langle \psi, Q^t g\rangle = 0 \text{ whenever } \psi \in \mathscr{D} \text{ and } \psi = 0 \text{ near } K. \tag{5}$$

But (5) is just the statement that $\mathrm{spt} Q^t g \subset K$. Thus condition (1) is equivalent to the statement that K is $Q^t - B^{-s}_{\infty,\infty\mathrm{cs}}$-null. Since the operator Q^t is elliptic of order $-m$, the corollary now follows from Corollary 2. ■

6. Some other 1-reductions

It is probably clear to the reader that the 1-reduction may be applied to good effect with spaces other than the T_β. To convert a problem about an operator S, acting on a space F, into a problem about 1, all that is necessary is that we be able to identify the space SF.

Take, for example, the scale of Sobolev spaces, $W^{k,p}$, for $k \in \mathbb{Z}_+$ and $1 < p < +\infty$. This scale may be imbedded, using Bessel potentials, in a full scale $W^{s,p}$, $s \in \mathbb{R}$. If S is an elliptic pseudodifferential operator of order m, then $S(W^{s,p}{}_{\mathrm{cs}})$ is locally-equivalent to $W^{s-m,p}$, and we are in business. We obtain the following theorem.

Theorem 13. *Under the conditions just stated, the following are equivalent, for a compact set* $K \subset \mathbb{R}^d$.

(1) *K is $S - W^{s,p}$-null.*

(2) *K is $1 - W^{s-m,p}$-null.*

If p' is the conjugate index to p, and Q is a parametrix for S, then the foregoing conditions are also equivalent to:

(3) *the space of all functions f that belong to $W^{-s,p'}$ and satisfy $Q^t f = 0$ near K are dense in $W^{-s,p'}$.*

For instance, this yields new results for singularities of holomorphic functions and approximation by holomorphic functions in $W^{2,p}$ norms, reducing these to problems whose solution is known.

Corollary 14. *Let $1<p\leqslant 2$ and $K\subset\mathbb{C}$ be compact. Then*:

(1) *the functions holomorphic near K are dense in $W^{2,p}$ if and only if K is $\bar\partial$–$\mathbf{L}^{p'}$-null;*

(2) *K is $\bar\partial$–$W^{2,p}$-null if and only if the functions holomorphic near K are dense in $\mathbf{L}^{p'}$.*

(The $\bar\partial$–$\mathbf{L}^q$-null sets referred to in (1) have been described by Carleson [3], and the sets in (2) were described by Bagby [1] (see also [7]).)

This approach also shows that harmonic approximation in $W^{1,2}$, on sets with no interior, is equivalent to holomorphic approximation in L^2, and hence is governed by the logarithmic capacity, and the classical fine topology of potential theory. In view of known results, this means that $W^{1,2}$ harmonic approximation (no interior) is equivalent to uniform harmonic approximation. In other words, if each function $\phi\in\mathscr{D}$ may be approximated in $W^{1,2}$ by functions harmonic near K, then it may be approximated uniformly by such functions, and conversely.

We may apply the 1-reduction to general Besov spaces $B^s_{p,q}$, provided we know that the operator S in question behaves well on the spaces in question. As noted above, it does not yet appear to be established that all smooth elliptic pseudodifferential operators, without restriction, behave well on all the intermediate ($p\neq 1$, ∞ or $s\notin\mathbb{Z}$) Besov spaces. There is no problem with the classical operators, like $\bar\partial$ or Δ, or indeed any properly-elliptic differential operator. When the 1-reduction applies, it reduces the problem of describing the $S-B^s_{p,q}$-null sets to that of describing the $1-B^{s-m}_{p,q}$-null sets, where $m=\text{order}S$. We give here a lemma which allows the reduced problem to be converted to equivalent forms.

Lemma 15. *Let F be a symmetric concrete space on $\mathbb{R}^d$ in which $\mathscr{D}$ is dense. Let K be a compact subset of $\mathbb{R}^d$. Then the following four conditions are equivalent*:

(1) *K is $1-F^*$-null;*

(2) *$\langle\phi, f\rangle=0$ whenever $f\in F^*$ has support in K, and $\phi\in\mathscr{D}$ satisfies $\phi=1$ near K;*

(3) *there is a net $\{\phi_\alpha\}\subset\mathscr{D}$ such that $\phi_\alpha=1$ near K and $\phi_\alpha\to 0$ in F-topology;*

(4) *$\{\phi\in\mathscr{D}:\phi=0$ near $K\}$ is dense in F.*

Remark. The hypotheses imply that F^* is an SCS.

PROOF. The annihilator of $\{\phi\in\mathscr{D}:\phi=0$ near $K\}$ in F^* is the set of those of $f\in F^*$ that are supported on K. By Hahn–Banach, (1) is equivalent to (4).

(4)⇒(3): Fix $\phi\in\mathscr{D}$, equal to 1 near K. There exist $\psi_\alpha\in\mathscr{D}$, equal to 0 near K, converging to ϕ in F-topology. Let $\phi_\alpha=\phi-\psi_\alpha$.

(3)⇒(2) : $\langle \phi, f \rangle = \langle \phi_\alpha, f \rangle \to 0$.

(2)⇒(1) : Fix $\phi \in \mathscr{D}$ with $\phi = 1$ near K. Let $f \in F^*$ have support in K. For each multi-index i, the distribution $x^i f$ belongs to F^* and has support in K. Thus

$$\langle x^i, f \rangle = \langle \phi, x^i f \rangle = 0.$$

Thus f annihilates the polynomials, hence annihilates $\mathscr{D}$, hence annihilates F, hence is 0. ■

In the formulation (3), and in the case of general Besov spaces, the problem of characterising such K has been studied by D. R. Adams. The space $B^s_{p,q}$ is, except in the cases $p = 1$ or $q = 1$, the dual of the closure of $\mathscr{D}$ in $F = B^{-s}_{p',q'}$. Adams has assembled and extended previous work to provide an alternative description of the capacities

$$\inf\{\|\phi\|_{B^t_{r,u}} : \phi \in \mathscr{D},\ \phi = 1 \text{ near } K\}$$

in terms of potentials, and has worked out the relation between these capacities and the Hausdorff measures. Details will appear in a book to be published by the Banach Centre, Warsaw. There is a preprint by Adams on this subject, 'The classification problem for the capacities associated with the Besov and Triebel–Lizorkin spaces' (Mathematics Department, University of Kentucky).

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REFERENCES

[1] Bagby, T. 1972 L^p approximation by analytic functions. *J. Approximation Theory* **5**, 401–4.

[2] Carleson, L. 1950 On null-sets for continuous analytic functions. *Arkiv för Math.* **1**, 311–18.

[3] Carleson, L. 1967 *Selected problems on exceptional sets*. Princeton, N.J. Van Nostrand.

[4] Dahlberg, B. E. J. 1977 On exceptional sets at the boundary for subharmonic functions. *Ark. Mat.* **15**, 305–12.

[5] Dolženko, E. P. 1963 The removability of singularities of analytic functions. *Uspekhi Mat. Nauk* **18**, 4 (112), 135–42.

[6] Harvey, F. R. and Polking, J. 1970 Removable singularities of solutions of linear partial differential equations. *Acta Math.* **125**, 39–56.

[7] Hedberg, L. I. 1972 Approximation in the mean by analytic functions. *Trans. Am. math. Soc.* **163**, 157–71.

[8] HÖRMANDER, L. 1985 *The analysis of linear partial differential operators*, vol. III. Berlin / Heidelberg. Springer.

[9] LEMARIE, P. G. 1984 *Algebres d'opérateurs et semi-groupes de Poisson sur un espace de nature homogène*. Thèse. Publications Mathematiques, Université de Paris-Sud, Centre d'Orsay.

[10] NGUYEN, X. UY 1979 Removable sets of analytic functions satisfying a Lipschitz condition. *Arkiv för Math.* **17**, 19–27.

[11] STEIN, E. M. 1970 *Singular integrals and differentiability properties of functions*. Princeton University Press.

[12] TAIBLESON, M. H. 1963 The preservation of Lipschitz spaces under singular integral operators. *Studia Math.* **24**, 107–11.

[13] TAIBLESON, M. H. 1964–6 On the theory of Lipschitz spaces of distributions on Euclidean n-space I. *J. Math. Mech.* **13** (1964), 407–80; II, *ibid.* **14** (1965), 821–40; III, *ibid.* **15** (1966), 973–81.

[14] TRIEBEL, H. 1976 Multipliers in Besov spaces and in L_p^{Ω}-spaces (the cases $0<p\leqslant 1$ and $p=\infty$). *Math. Nachr.* **75**, 229–45.

[15] TRIEBEL, H. 1978 On spaces of $B^s_{\infty,q}$ type and C^s type. *Math. Nachr.* **85**, 75–90.

[16] TRIEBEL, H. 1983 *Theory of function spaces*. Basel/Boston/Stuttgart. Birkhäuser.

[17] VERDERA, J. 1987 C^m approximation by solutions of elliptic equations, and Calderon–Zygmund operators. *Duke Math. J.* **55**, 157–87.

[18] ZYGMUND, A. 1945 Smooth functions. *Duke Math. J.* **12**, 47–76.