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Rational Approximation and Weak Analyticity. I

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1

This paper is about rational approximation in the uniform norm on plane compact sets. In subsequent papers we propose to deal with some other norms.

We are interested in the extent to which the approximability of a function f by rationals on a compact set X can be characterised in terms of some kind of "weak analyticity" of f on X. By "weak analyticity" of f on X we mean that a suitably-interpreted $\bar{\partial}$-derivative of f should vanish on some suitable subset of X. This vague notion seems a priori reasonable, and motivated early work on rational approximation.

Let $C(X)$ denote the uniform algebra [G 1] of all continuous complex-valued functions on X, and let $R(X)$ denote the closure in $C(X)$ of the space of all functions which are holomorphic in a neighbourhood of X. We regard functions $f \in C(X)$ as functions on $\mathbb{C}$, by extending them in any continuous manner to $\mathbb{C}$. The following is known as Vitushkin's individual function theorem [V]:

A given function $f \in C(X)$ belongs to $R(X)$ if and only if there exists $\kappa > 0$ such that

$$\left| \int f \frac{\partial \phi}{\partial \bar{z}} d\mathscr{L}^2 \right| \leqq \kappa d \|\nabla \phi\|_\infty \gamma(D \sim X)$$

whenever $\phi \in \mathscr{D}$, D is a disc containing spt ϕ, and $d = \operatorname{diam} D$.

Here, $\mathscr{L}^2$ denotes Lebesgue measure on $\mathbb{C}$, γ denotes analytic capacity, and $\mathscr{D}$ is the Schwartz space of test functions. This necessary and sufficient condition may, of course, be viewed as a kind of weak analyticity on X. We are interested in stronger kinds, involving the existence of a function ∇f. Our first result is as follows.

Theorem 1. *Suppose X is a compact subset of $\mathbb{C}$, $2 < p \leqq \infty$, and f belongs to the Sobolev space $W^{1;p}_{\text{loc}}$. Then $f \in R(X)$ if and only if $\frac{\partial f}{\partial \bar{z}}(a) = 0$ at $\mathscr{L}^2$ almost all nonpeak points a.*

Here, the space $W^{1,p}_{\text{loc}}$ consists of all functions $f \in L^p_{\text{loc}}$ whose first distributional derivative also belongs to L^p_{loc}. (In particular, the result applies to smooth f and to $f \in \text{Lip}\,1$.) The concept of *peak point* comes from the theory of uniform algebras, and was originally motivated by the notion of *barrier* in potential theory. The point a is a peak point iff there exists $g \in R(X)$ such that $|g(z)| < 1 = g(a)$ whenever $a \neq z \in X$. It is helpful to think of the set $Q(X)$ of nonpeak points as a kind of interior of X [although we do not know for sure whether always $Q(X \cap Y) \subset Q(X) \cap Q(Y)$]. Nonpeak points have been described in terms of analytic capacity by Melnikov [M]: $a \in X$ is a nonpeak point iff

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n(a) \sim X) < +\infty,$$

where $A_n(a)$ denotes the annulus

$$\{z \in \mathbb{C} : 2^{-n-1} < |z-a| < 2^{-n}\}.$$

There are other characterisations, and some useful sufficient conditions [G 1, (VIII.4)], and it is often possible to identify the nonpeak points in examples. Even if that cannot readily be done, Theorem 1 can be applied as long as we can identify a subject having full measure in $Q(X)$. By Vitushkin's instability theorem [V], such a subset is

$$\left\{a \in X : \lim_{r \downarrow 0} \frac{\gamma(B(a,r) \sim X)}{r^2} = 0\right\},$$

where $B(a, r)$ denotes the closed disc having centre a and radius r. By way of further clarification, we remark that all points of the *outer boundary* of X (the union of the boundaries of the components of $\mathbb{C} \sim X$) are peak points, so the condition places no restriction on f on the outer boundary, even if it has positive area. The condition $\frac{\partial f}{\partial \bar{z}} = 0$ a.e. on U, where U is open and $f \in W^{1,p}_{\text{loc}}$, implies that f is holomorphic on U, so we may rephrase the condition of Theorem 1 as:

> *f is holomorphic on* $\text{int}\,X$ *and* $\frac{\partial f}{\partial \bar{z}} = 0$ *at almost all points a of the inner boundary of X at which*
>
> $$\lim_{r \downarrow 0} \frac{\gamma(B(a,r) \sim X)}{r^2} = 0.$$

(Of course, the inner boundary is the complement in *bdy* X of the outer boundary.)

For smooth f, Theorem 1 may be deduced from Browder's theorem [B, p. 166 (3.2.9)], since the "essential set" used there is in fact the closure of the set Q of nonpeak points. A number of other special cases have been noted in the interim [R, OF 2]. Notably, Khavinson [K] proved that for sets S having "finite perimeter" and for $f \in \text{Lip}\,1$, f belongs to $R(X)$ if and only if $\frac{\partial f}{\partial \bar{z}} = 0$ a.e. on X. For general X, this condition is not necessary, even for $f \in \text{Lip}\,1$. Sets of finite perimeter, such as Swiss cheeses, are very special, in that the set of peak points in X not only has area zero, but has Hausdorff dimension one.

Our second result shows weak differentiability of *general* $f \in R(X)$. Previous work [OF 3, OF 4, W 1, W 2] showed that the functions in $R(X)$ are "all but" differentiable at $\mathscr{L}^2$ almost all nonpeak points. Specifically, Wang [W 2] showed that for $\mathscr{L}^2$ almost every $a \in Q$ and each $\alpha \in (0, 1)$, there is a set E, having full area density at a, and a constant $\kappa > 0$ such that

$$|f(z) - f(a)| \leqq \kappa |z - a|^{\alpha} \|f\|_{R(X)}$$

whenever $f \in R(X)$ and $z \in E$. Wermer's example [WE] of an X such that $R(X)$ admits no bounded point derivation rules out the possibility that this could work for $\alpha = 1$. However, it leaves open the possibility of some kind of a.e. differentiability of f, and this is what we obtain. Given a measurable set E, let $D^{1,p}(E)$ denote the space of $f \in L_p(E)$ for which there exist $f_1, f_2 \in L_p(E)$ such that for $\mathscr{L}^2$ almost all $b \in E$,

$$\frac{1}{r^3} \int_{B(b,r)} |f(z) - f(b) - (z - b) f_1(b) - \overline{(z - b)} f_2(b)| \, d\mathscr{L}^2(z) \to 0$$

as $r \downarrow 0$. For such f, we denote f_1 by $\dfrac{\partial f}{\partial z}$ and f_2 by $\dfrac{\partial f}{\partial \bar{z}}$.

Theorem 2. *Let $f \in R(X)$. Let a be a nonpeak point, and $1 \leqq p < 2$. There exists a set E of nonpeak points, having full area density at a, such that $f \in D^{1,p}(E)$, and $\dfrac{\partial f}{\partial \bar{z}} = 0$ a.e. on E.*

It is of interest to relate Theorems 1 and 2 to the theory of finely holomorphic functions. Let $f: U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is finely open. Then the following are equivalent [L, F]:

(i) *f is finely holomorphic on U;*

(ii) *Each $a \in U$ has a Euclidean-compact fine neighbourhood X such that $f \in R(X)$;*

(iii) *Each $a \in U$ has a Euclidean-compact fine neighbourhood X such that $f \in \mathrm{Lip}(1, X)$ and $\dfrac{\partial f}{\partial \bar{z}} = 0$ on X.*

Clearly this result is closely related to Theorem 1. In fact, a preliminary version of Theorem 1 was used to prove it in [L]. However, there are sets X with empty fine interior and lots of nonpeak points, so neither Theorem 1 nor Theorem 2 can be inferred from the theory of fine holomorphy. Furthermore, *there is no possibility* that for general X and general continuous f we could characterise $f \in R(X)$ by a local condition at the nonpeak points. Davie's example [D 2] provides a set X such that the only nonpeak points are the interior points of X, yet there exists a function $f \in C(X)$, *analytic* on $\operatorname{int} X$, yet not belonging to $R(X)$.

In connection with Theorem 2, Steen's cheese [ST] is relevant. It has an $f \in R(X)$ for which $\dfrac{\partial f}{\partial \bar{z}} = \dfrac{\partial f}{\partial z}$ and is a (nonzero) measure, singular to $\mathscr{L}^2$. In fact, $\dfrac{\partial f}{\partial \bar{z}}$ is $\mu \times \mathscr{L}^1$, where μ is $(\log_3 2)$-dimensional Hausdorff measure on the Cantor set C. The set $C \times \mathbb{R}$ certainly has some nonpeak points, but of course it has zero area density at all points.

Vitushkin's individual function theorem remains the most explicit necessary and sufficient condition for a general continuous f to belong to $R(X)$.

2. Proofs

(2.1) First we prove sufficiency in Theorem 1.

Suppose $\frac{\partial f}{\partial \bar{z}}=0$ at $\mathscr{L}^2$ almost all points of Q. Let μ be a Borel measure on X, annihilating $R(X)$. Then the Cauchy transform $\hat{\mu}$, defined by

$$\hat{\mu}(z)=\frac{1}{\pi}\int\frac{d\mu(w)}{z-w},$$

is supported on X, and belongs to $L_q(\mathscr{L}^2)$, where q is the conjugate index to p. For $\phi\in\mathscr{D}$ we have

$$\int\phi d\mu=\int\frac{\partial\phi}{\partial\bar{z}}\cdot\hat{\mu}d\mathscr{L}^2,$$

so by continuity, this continues to hold for $\phi\in W^{1;p}_{\mathrm{loc}}$. Now $\hat{\mu}=0$ at $\mathscr{L}^2$ almost all peak points [G 1, (II.11.4), p. 54], so

$$\int f d\mu=\int\frac{\partial f}{\partial\bar{z}}\cdot\hat{\mu}d\mathscr{L}^2=0.$$

By the separation theorem, $f\in R(X)$.

(2.2) Next, we prove necessity in Theorem 1. For this implication, we need only that $f\in W^{1,1}_{\mathrm{loc}}$.

Suppose $f\in R(X)$. By Vitushkin's individual function theorem, there exists $\kappa_1>0$ such that

$$\left|\int f\frac{\partial\phi}{\partial\bar{z}}d\mathscr{L}^2\right|\leqq\kappa_1 d\|\nabla\phi\|_\infty\gamma(D\sim X)$$

whenever $\phi\in\mathscr{D}$, D is a disc containing *spt* ϕ, and $d=\operatorname{diam}D$. Since $f\in W^{1,1}_{\mathrm{loc}}$, we have

$$\int f\cdot\frac{\partial\phi}{\partial\bar{z}}d\mathscr{L}^2=-\int\phi\cdot\frac{\partial f}{\partial\bar{z}}d\mathscr{L}^2.$$

Now take $\phi_n\in\mathscr{D}$ and discs D_n of diameter d_n, such that $\int\phi_n d\mathscr{L}^2=1$, spt $\phi_n\subset D_n$, $d_n\downarrow 0$, $d_n^2\|\phi_n\|_\infty\leqq\kappa_2<+\infty$, and $d_n^3\|\nabla\phi_n\|_\infty\leqq\kappa_3<+\infty$. Then on the Lebesgue set of $\frac{\partial f}{\partial\bar{z}}$, we have

$$\frac{\partial f}{\partial\bar{z}}*\phi_n\to\frac{\partial f}{\partial\bar{z}},$$

pointwise, where $*$ denotes convolution. Let a be a point of the Lebesgue set of $\frac{\partial f}{\partial\bar{z}}$ at which

$$\frac{\gamma(B(a,r)\sim X)}{r^2}\to 0$$

as $r \downarrow 0$. By Vitushkin's instability theorem and Curtis' criterion [G 1, p. 204], $\mathscr{L}^2$ almost all nonpeak points are of this type. Then

$$d_n \|\nabla\phi_n(a+\cdot)\|_\infty \gamma((a+D_n)\sim X)\to 0$$

as $n\uparrow\infty$, so we conclude that $\frac{\partial f}{\partial\bar z}(a)=0$. This proves the result.

(2.3) We now prove Theorem 2.

Fix a nonpeak point $a\in X$, a function $f\in R(X)$, and $p\in[1,2)$. To prove the result, we have to produce a set E, having full density at a, and a function $h\in L_p(\mathscr{L}^2, E)$ such that

$$\lim_{r\downarrow 0}\frac{1}{r^3}\int |f(z)-f(b)-(z-b)h(b)|\,d\mathscr{L}^2 z=0 \tag{1}$$

for $\mathscr{L}^2$ almost all $b\in E$. If f were smooth, the function h would be $\frac{\partial f}{\partial z}$.

If μ is a measure annihilating $R(X)$, then

$$f\hat\mu=\widehat{f\mu}$$

$\mathscr{L}^2$ almost everywhere. Formally, we obtain

$$\hat\mu\frac{\partial f}{\partial z}+fB\mu=B(f\mu), \tag{2}$$

where B is the Beurling transform [A, Chap. V], defined for test functions by $B\phi(z) =\frac{1}{\pi}PV\int\frac{\phi(w)}{(w-z)^2}\,d\mathscr{L}^2 w$.

The formula (2) is the key idea. All we have to do is find a μ for which $B\mu$ and $B(f\mu)$ exist a.e., and $\hat\mu$ is nonzero on a set having full density at a, and then (2) gives us the desired function h, namely

$$h=(\hat\mu)^{-1}\{B(f\mu)-fB(\mu)\}.$$

By a theorem of Davie [D 1; G 2, p. 132], a has a representing measure $\nu\in L_1(\mathscr{L}^2, Q)$, where Q is the set of nonpeak points. Replacing μ by $(z-a)\nu$, we obtain

$$\hat\mu=-\frac{1}{\pi}+(z-a)\hat\nu,$$

$$B\mu=\hat\nu+(z-a)B\nu,$$

$$\widehat{f\mu}=-\frac{f(a)}{\pi}+(z-a)\widehat{f\nu},$$

$$B(f\mu)=\widehat{f\nu}+(z-a)B(f\nu).$$

By the Calderon-Zygmund theory [S, p. 42, Theorem 4; A] $B\nu$ and $B(f\nu)$ are weak-type L_1. In particular, they exist $\mathscr{L}^2$ a.e. By [OF 1, Lemma 2, p. 406], the set $E_1=\{z: |(z-a)\hat\nu(z)|<1/2\pi\}$ has full area density at a. Since $|\hat\mu|>\frac12$ on E_1, the function

$$h=(\hat\mu)^{-1}\{\widehat{f\nu}+(z-a)B(f\nu)-f\hat\nu-(z-a)B\nu\}$$

is properly-defined $\mathscr{L}^2$ a.e. on E_1.

It remains to construct a set $E\subset E_1$, having full area density at a with $\int_E |h|^p d\mathscr{L}^2 < +\infty$. Pick $\alpha>0$ with $p(1+\alpha)<2$. For $n=1,2,3,\ldots$, let

$$A_n=\{z\colon 2^{-n-1}\leqq|z-a|<2^{-n}\},$$

$$G_n=\{z\in A_n\colon |Bv|\leqq 2^{(2+\alpha)n},\ |B(fv)|\leqq 2^{(2+\alpha)n}\},$$

$$E_2=\bigcup_{n=1}^{\infty} G_n,$$

$$E=E_1\cap E_2.$$

Since Bv and $B(fv)$ are weak-type L_1, there exist constants $\kappa_1>0$ and $\kappa_2>0$ such that

$$\mathscr{L}^2\{|Bv|\geqq\lambda\}\leqq\kappa_1/\lambda$$

$$\mathscr{L}^2\{|B(fv)|\geqq\lambda\}\leqq\kappa_2/\lambda,$$

for all $\lambda>0$. Thus $\mathscr{L}^2(A_n\sim G_n)\leqq\kappa_3 2^{-\alpha n}\mathscr{L}^2(A_n)$ whence E_2, and hence E, have full area density at a. Also,

$$\int_{E_2}|(z-a)Bv|^p d\mathscr{L}^2=\sum_{n=1}^{\infty}\int_{G_n}|(z-a)Bv|^p d\mathscr{L}^2$$

$$\leqq\sum_{n=1}^{\infty}\kappa_1 2^{(1+\alpha)pn}\mathscr{L}^2(A_n)<+\infty,$$

and similarly,

$$\int_{E_2}|(z-a)B(fv)|^p d\mathscr{L}^2<+\infty.$$

Since $\widehat{fv}$ and $\hat{v}$, like all Cauchy transforms of measures, are locally p-th power integrable, we conclude that

$$\int_E |h|^p d\mathscr{L}^2<+\infty,$$

as required.

3. Concluding Remarks

(3.1) **Corollary to Theorem 1.** *Let $f\in W^{1,p}_{\mathrm{loc}}$ for some $p>2$, and let A be countable. Suppose that $f^{-1}(A)$ is closed, and each $a\in X\sim f^{-1}(A)$ has a closed neighbourhood N such that $f\in R(N\cap X)$. Then $f\in R(X)$.*

For example, if $f\in W^{1,p}_{\mathrm{loc}}$ for some $p>2$, and f belongs to $R(X)$ locally off the zero set of f, then $f\in R(X)$. In particular, if $f\in W^{1,p}_{\mathrm{loc}}$ and $f^2\in R(X)$, then $f\in R(X)$. It is an open question whether $f\in C(X)$ and $f^2\in R(X)$ imply $f\in R(X)$. Paramonov [P] showed that if $f\in \mathrm{Lip}(\alpha,X)$ for some $\alpha>\frac{2}{3}$ and f belongs to $R(X)$ locally off its zero set, then $f\in R(X)$. Since $W^{1,p}_{\mathrm{loc}}\subset\mathrm{Lip}(\alpha,X)$ for $\alpha<\dfrac{p-2}{p}$, Paramonov's result leads to the same conclusion as ours for $p>6$.

More generally, the Corollary shows that if $f \in W^{1,p}_{\mathrm{loc}}$ and $p(f) \in R(X)$ for some polynomial p, then $f \in R(X)$ (Take $A = (p')^{-1}(0)$). Indeed, this works even for p analytic on a neighbourhood of $f(X)$.

(3.2) Assuming Theorem 2, it is possible to give another proof of necessity in Theorem 1. The space $W^{1,p}_{\mathrm{loc}}$ is a subset of $D^{1,p}(E)$, for each bounded E, and the distributional and pointwise derivatives agree a.e.

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