LNM 995 (1983) 103-122

QUALITATIVE RATIONAL APPROXIMATION ON PLANE COMPACTA

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Abstract

Let X be a compact subset of the complex plane. Let $R(X)$ denote the space of all rational functions with poles off X. Let $A(X)$ denote the space of all complex-valued functions on X that are analytic on the interior of X. Let $\mathcal{A}(X)$ be a Banach space of functions on X, with $R(X) \subset \mathcal{A}(X) \subset A(X)$. Consider the problems: (1) Describe the closure of $R(X)$ in $\mathcal{A}(X)$. (2) For which X is $R(X)$ dense in $\mathcal{A}(X)$? There are many results on these problems, for various particular Banach spaces $\mathcal{A}(X)$. We offer a point of view from which these results may be viewed systematically.

1. Introduction.

(1.1) Let me begin by indicating how my topic fits into the world of mathematics.

Qualitative rational approximation theory answers the question: is it possible to approximate a given function arbitrarily closely by rationals? There is also a quantitative theory, which addresses the question: how closely can a given function be approximated by rational functions of a specified degree? Experience indicates that the quantitative theory lags behind the qualitative by anything up to thirty years. For the kind of qualitative results I shall present, the quantitative theory is nonexistent or primitive.

After the quantitative theory comes the computational theory, culminating in practical computer programs. The expected time lag here is another twenty years or thereabouts so perhaps by 2020 A.D. the results will be ready for use by the engineers and scientists. What

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will the engineers and scientists do with them? It is probably better not to know. For instance, the theorem of Weierstrass, that periodic functions on the line may be approximated uniformly by trigonometric polynomials, finds application in digital recording. This makes possible the indefinite preservation, with effectively complete fidelity, of all kinds of unspeakable rubbish.

I shall not deal explicitly with polynomial approximation. Except in the case of L^p approximation $(1 \leq p < +\infty)$, polynomial approximation is possible if and only if rational approximation is possible and the set is polynomially-convex. For L^p polynomial approximation there are very interesting and formidable problems. See [2] for results and references to the work of M. M. Džrbašjan, V. P. Havin, V. G. Mazja, S. N. Mergelyan, A. P. Tamadjan, A. L. Šaginjan, and S. O. Sinanjan.

I restrict myself to compact sets. For weighted uniform approximation the theory on closed unbounded sets has been developed by N. U. Arakelian, P. M. Gauthier, W. Hengartner, A. Roth, S. Scheinberg, J. L. Walsh, and others. See [7,8] for references.

The theory in several variables is comparatively primitive.

(1.2) My objective is to give the essential facts about qualitative rational approximation on plane compacta. I shall endeavour to make the results seem intelligible and natural, but I will not include any proofs. Suffice it to say that the proofs are quite inhomogeneous, and in many cases long and intricate.

My account will not be historical, nor will it reflect the logical structure of the existing theory. Instead, I shall present the results in an order in which I would <u>like</u> to be able to prove them, from my point of view.

The results I shall present were developed in the period 1955-1980. The main contributers to the core of the theory were T. Bagby, A. Browder, L. Carleson, A. M. Davie, E. P. Dolženko, V. P. Havin,

S. Ya. Havinson, J. Garnett, A. A. Gonchar, L. I. Hedberg, M. S. Melnikov, S. N. Mergelyan, S. O. Sinanjan, A. G. Vitushkin, and J. Wermer. In the prehistory of the subject, during the first seventy years after the Weierstrass approximation theorem, the outstanding contributers were C. Runge, H. Lebesgue, J. L. Walsh, T. Carleman, A. Roth, O. J. Farrell, F. Hartogs, A. Rosenthal, and M. V. Keldysh.

(1.3) Arbitrary plane compacta can be pretty complicated objects, and as a result, rational approximation on such sets is an extremely perverse subject, teeming with surprising counterexamples. I am not going to dwell on examples. I am going to present positive results. In doing this, I run the risk that the beginner may not appreciate that the most surprising thing about the subject is the existence of a moderately extensive body of positive results; and consequently that he may feel that the formulation of some of these results is a little complicated. He can rest assured that all the simple answers have been tried and found wanting.

§2. Formulation of the problems.

(2.1) Let X be an arbitrary compact subset of the complex plane, $\mathbb{C}$. Let $\mathcal{R}(X)$ denote the space of all rational functions with poles off X. Let $\mathcal{A}(X)$ denote the space of all functions on X, analytic on the interior of X. Let $B(X)$ be one of a certain list (see (2.2) below) of Banach spaces of complex-valued functions on X, such that $\mathcal{R}(X) \subset B(X)$ and the subspace $A(X) = B(X) \cap \mathcal{A}(X)$ is closed. Let $R(X)$ denote the closure of $\mathcal{R}(X)$ in $B(X)$. Then $R(X) \subset A(X)$. We consider the following two main problems:

(1) Give a reasonable description of $R(X)$, i.e. give an explicit condition on a function $f \in B(X)$, necessary and sufficient for $f \in R(X)$.

(2) For which compact X is $R(X) = A(X)$?

(2.2) The main examples of Banach function spaces $B(X)$ are as follows.

(1) $L^p(X) = L^p(X,m)$ $(1 \leq p < \infty)$, where m is area measure on X.

(2) $C(X)$, the space of continuous functions on X, with the sup norm, $\|f\|_\infty$.

(3) $\mathrm{Lip}(\alpha,X)$ $(0 < \alpha \leq 1)$, the space of functions f for which there exists a constant $\kappa > 0$ such that

$$|f(x) - f(y)| \leq \kappa|x-y|^\alpha$$

for all $x,y \in X$. The norm of f is $\|f\|_\infty$ plus the least value of κ.

(4) $C^k(X)$, for k a positive integer. This space has two versions. Let C^k denote the space of k-times continuously-differentiable functions on $\mathbb{C}$, with bounded partials up to order k. Then C^k forms a Banach algebra with the norm

$$\|f\| = \sum_{j=o}^{k} \sum_{r+s=j} \frac{r!s!}{j!} \left\| \frac{\partial^j f}{\partial x^r \partial y^s} \right\|_\infty .$$

Let I and J denote the closed ideals

$I = \{f \in C^k : f = 0 \text{ on } X\}$,

$J = \{f \in C^k : \dfrac{\partial^j f}{\partial x^r \partial y^s} = 0 \text{ on } X,\ 0 \leq j \leq k,\ r + s = j\}$.

The <u>function version</u> of $C^k(X)$ is the quotient space C^k/I (with the quotient norm). The <u>jet version</u> of $C^k(X)$ is the quotient space C^k/J. The jet version has an alternative, local, description, via Whitney's extension theorem [23, Ch. 6]. The natural quotient map from the jet version onto the function version is occasionally injective and a homeomorphism, but usually not. In case $k = 1$, the function version has a local description [13].

(5) $\mathrm{Lip}(k+\alpha, X)$, $0 < k \in \mathbb{Z}$, $0 < \alpha \leq 1$. This space has a function version and a jet version, derived from the global space $\mathrm{Lip}(k+\alpha)$ of functions in C^k with k-th partial derivatives in $\mathrm{Lip}\ \alpha$. There is a local description of the space [23, Ch. 6, p. 176].

(6) The Sobolev spaces $W^{k,p}(X)$, $0 < k \in \mathbb{Z}$, with function and jet versions, derived from the global spaces $W^{k,p}$ of functions whose k-th distribution derivatives are representable by L^p functions.

Apart from these main examples, there are weighted L^p spaces, mean Lipschitz spaces, weighted Sobolev spaces, and so on.

Note that in every instance B is really a function $X \mapsto B(X)$ on the compact subsets of $\mathbb{C}$. Moreover, in each instance $B(X)$ may be derived from a global space $B = B(\mathbb{C})$ by restriction, and the norm induced by $B(\mathbb{C})$ is equivalent to the given norm. Thus $B(F)$ may be defined by restriction for all closed $F \subset \mathbb{C}$. Each B has a <u>localness</u> property: if X is compact, $f : X \to \mathbb{C}$, and each point $a \in X$ has a closed relative neighbourhood Y in X such that $f \in B(Y)$, then $f \in B(X)$. Each B has a technical property, called T-<u>invariance</u>. It states that if $f \in B(X)$ and $\phi \in \mathcal{D}$(= the space of C^∞ functions with compact support), then the function $T_\phi f$ defined by

$$T_\phi f = \phi f - \widehat{f \frac{\partial \phi}{\partial \bar{z}} m}$$

also belongs to $B(X)$, where the <u>Cauchy transform</u> $\hat{\mu}$ of a measure μ is defined by

$$\hat{\mu}(z) = \frac{1}{\pi} \int \frac{d\mu(\zeta)}{z-\zeta} .$$

The operator T_ϕ is called the <u>Vitushkin localization operator</u>. It is used to chop up the singular set of an analytic function.

(2.3) The above examples of Banach function spaces $B(X)$ are ordered by continuous (and usually compact) inclusion maps:

$$W^{k+1,p}(X) \to C^k(X) \to W^{k,p}(X) \quad ,$$

$$\mathrm{Lip}(k+\alpha,X) \to C^k(X) \to \mathrm{Lip}(k-1+\alpha,X) \quad ,$$

$$\mathrm{Lip}(\alpha,X) \to C(X) \to L^p(X) \quad ,$$

$$L^p(X) \to L^{p'}(X) \quad , \quad (p > p') \quad ,$$

$$W^{k,p}(X) \to W^{k,p'}(X) \quad , \quad (p > p') \quad ,$$

$$W^{k+1,p}(X) \to \mathrm{Lip}(k+\frac{1}{q},X) \quad , \quad (\frac{1}{p}+\frac{1}{q} = 1) \quad ,$$

$$\mathrm{Lip}(k+\alpha,X) \to \mathrm{Lip}(k+\alpha',X) \quad , \quad (\alpha > \alpha) \quad .$$

Approximation in one space implies approximation in all larger spaces, since the injections are continuous.

§3. Preliminary.

(3.1) Rational approximation in all these Banach spaces is local. Precisely speaking, we have the following result, due to E. Bishop for $B = C$.

Theorem. *Let* $f \in B(X)$, *and suppose every point* $a \in X$ *has a closed relative neighbourhood* $Y \subset X$ *such that* $f|Y \in R(Y)$. *Then* $f \in R(X)$.

This means that the answers to the main questions (1) and (2) of (2.1) should involve only local properties of f and X. The proof is an application of the localization operator, as in [14].

(3.2) Theorem (Runge). Let $f \in B(X)$ be (the restriction to X of a function) analytic on a neighbourhood of X. Then $f \in R(X)$.

See [15]. As a result of this, the closure of $R(X)$ in $B(X)$ is the same as the closure of

$$H(X) = \{f : f \text{ is analytic on a neighbourhood of } X\}.$$

(3.3) From the C^1 level up, i.e. as soon as $B \hookrightarrow C^1$ is continuous, the point Cauchy-Riemann operator (at a point $a \in X$):

$$\bar{\partial}\cdot(a) : f \to \bar{\partial}f(a) = \tfrac{1}{2}\left\{\frac{\partial f}{\partial x}(a) + i\,\frac{\partial f}{\partial y}(a)\right\}$$

can be expected to play a rôle. It will be a continuous linear functional on $B(X)$. Since $\bar{\partial}f(a) = 0$ for $f \in R(X)$, it follows by continuity that $\bar{\partial}f(a) = 0$ for $f \in R(X)$.

(3.4) Since $R(X)$ consists of functions which are infinitely-differentiable on a neighbourhood of X, the closure of $R(X)$ in $B(X)$ is contained in the closure of C^∞ in $B(X)$. Thus it makes little sense to work with a space $B(X)$, in which C^∞ is not dense. This is why $L^\infty(X)$ was left out of the list in (2.2); the closure of C^∞ in $L^\infty(X)$ is $C(X)$. For the same reason we must work with the spaces $\mathrm{lip}(k+\alpha,X)$ instead of $\mathrm{Lip}(k+\alpha,X)$, for $0 < \alpha < 1$. These spaces are the closures of C^∞ in the respective norms. For $0 < \alpha < 1$, $\mathrm{lip}(\alpha,X)$ is the space of functions $f : X \to \mathbb{C}$ such that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \le \varepsilon|x-y|^\alpha$$

whenever $x \in X$, $y \in X$, and $|x-y| < \delta$. It is also the space of restrictions of the corresponding global $\mathrm{lip}\,\alpha$ space [23]. For $0 < k \in \mathbb{Z}$ and $0 < \alpha < 1$, $\mathrm{lip}(k+\alpha,X)$ is the space (with two versions) of functions (or jets) on X with extensions in global $\mathrm{lip}(k+\alpha)$, which in turn is the space of functions $f : \mathbb{C} \to \mathbb{C}$ such that f and all partial derivatives up to order k belong to $\mathrm{lip}\,\alpha$. For $0 < k \in \mathbb{Z}$, the closure of C^∞ in $\mathrm{Lip}(k,X)$ is not so easily described, but for $k = 1$, it is known [16] that

$$f \in \operatorname{clos}_{\mathrm{Lip}(1,X)} R(X)$$

if and only if

$$f \in \operatorname{clos}_{C^1(X)} R(X),$$

i.e. Lip 1 rational approximation reduces to C^1 rational approximation; this renders it unnecessary to describe the Lip 1 closure of C^∞ on X. It may be possible to do the same thing for Lip k.

(3.5) In general, an optimist might hope that the answer to main problem (2) would be categorically appropriate to the functor B, and would be free of "analytic" elements, i.e. that for $B = L^p$ it would involve only area, for $B = C$ it would be topological, for $B = \mathrm{Lip}\,\alpha$ it would be metric, for $B = C^1$ it would involve the C^1-differential structure of X, and so on. As we shall see, some of these hopes are fulfilled.

(3.6) The outline of the theory is as follows. The L^p theory, for $1 \le p < 2$, is trivial. The remaining B divide into two broad classes: the <u>smooth</u> class $(B \hookrightarrow C^1)$, and the <u>hairy</u> class $(C^1 \hookrightarrow B\ ,\ B \neq C^1)$. In the smooth class the point Cauchy-Riemann operator plays a crucial role, and the results are pretty simple to state. In the hairy class, which embraces $L^p(2 \le p < \infty)$, C, and $\mathrm{lip}\,\alpha(0 < \alpha < 1)$, it takes a bit of effort to digest the statements, let alone the proofs.

(3.7) There are several unsolved problems in the theory, and I have indicated some of them as they arise.

4. The Smooth Class.

(4.1) Consider the jet spaces $B = C^k$, $\mathrm{lip}(k+\alpha)$. All jets in $R(X)$ satisfy the Cauchy-Riemann equations on X. The jets in $A(X)$ need only satisfy the Cauchy-Riemann equations on the closure of the interior of X. For any point $a \in X \sim \mathrm{clos\ int}\ X$, there exists $f \in A(X)$ such that $\bar{\partial} f(a) \neq 0$. Thus, if $X \neq \mathrm{clos\ int}\ X$, then $R(X) \neq A(X)$.

The following theorem solves main problem (1) for the jet space $B = \mathrm{lip}(k+\alpha)$ [17].

Theorem. *Let $0 < k \in \mathbf{Z}$, $0 < \alpha < 1$, $B = \mathrm{lip}(k+\alpha)$, and let $X \subset \mathbf{C}$ be compact. Then $R(X)$ is the set of all jets $f \in B(X)$ such that*

$$\frac{\partial^j \bar{\partial} f}{\partial x^r \partial y^s} = 0$$

on X for $0 \leq j \leq k-1$, $r + s = j$.

This says that a jet is approximable by rationals if and only if it satisfies the Cauchy-Riemann equations on X, and also satisfies all consequences of the Cauchy-Riemann equations which make sense for jets in $B(X)$.

As an immediate corollary, we have the solution of main problem (2) for $B = \mathrm{lip}(k+\alpha)$.

Corollary. $R(X) = A(X)$ if and only if $X = \mathrm{clos\ int}\ X$.

The corresponding results for C^k and $W^{k,p}$ have not yet been established.

(4.2) For the *function* spaces C^k and $\mathrm{lip}(k+\alpha)$, problem (1) has not been solved. Problem (2) has been solved only for sets with empty interior.

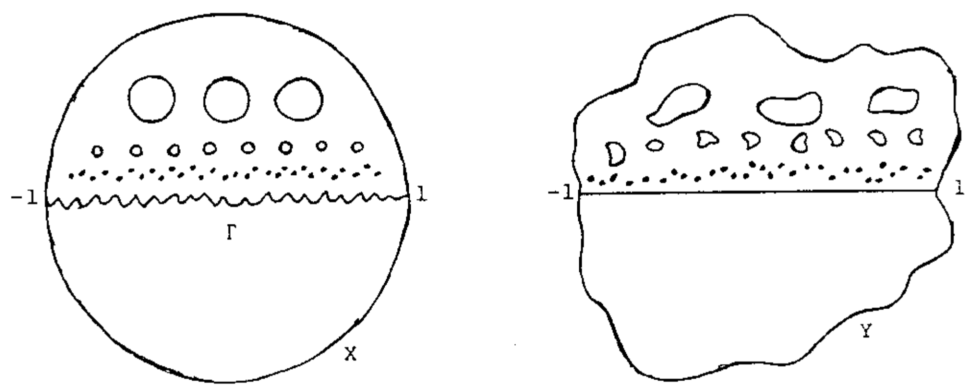
<u>Theorem</u> [18,17]. <u>Let</u> $B = C^k$ or $\mathrm{lip}(k+\alpha)$, $0 < k \in \mathbf{Z}$, $0 < \alpha < 1$. <u>Then</u> $R(X) = B(X)$ <u>if and only if</u> X <u>is a subset of a finite union of pairwise-disjoint simple curves of class</u> B (i.e. class C^k if $B = C^k$ and class $\mathrm{lip}(k+\alpha)$ if $B = \mathrm{lip}(k+\alpha)$).

The corresponding result for $W^{k,p}$ has not been proved.

(4.3) I remark parenthetically that Theorem (4.1) yields the solution of the rational approximation problems in the Fréchet jet space $C^\infty(X)$, since $C^\infty(X)$ is the injective limit of the spaces $\mathrm{lip}(k+\frac{1}{2},X)$.

§5. <u>An example</u>.

Before presenting the results on the hairy class of B's, I give an example to indicate the flavour.



<u>Figure 1</u>

Consider the open unit disc D, and let Γ be an arc of positive area joining -1 to 1 and otherwise lying in D. Take a countable family of open discs $D_n \subset D \setminus \Gamma$, with $\sum_n \mathrm{diam}\, D_n < +\infty$, such that each point of Γ belongs to $\mathrm{clos} \bigcup_n D_n$. Let

$$X = \text{clos } D \sim \bigcup_{n=1}^{\infty} D_n .$$

Find a homeomorphism ϕ of $\mathbb{C}$ onto $\mathbb{C}$ which maps Γ onto the line segment $[-1,1]$. Let $Y = \phi(X)$. Then it can be shown [cf. 5, p. 220 and p. 235 (13.2)] that for $B = C$ we have $R(X) \neq A(X)$ and $R(Y) = A(Y)$. Thus uniform rational approximation is not determined by topological criteria.

§6. Second main problem for the hairy class.

(6.1) For $1 \leq p < 2$, the L^p problems have a trivial answer.

Theorem. *Let* $1 \leq p < 2$, $B = L^p$. *Then* $A(X) = R(X)$ *for all compact* X.

(6.2) Consider $B = L^p$ $(2 \leq p < \infty)$, C, or lip $\alpha(0 < \alpha < 1)$. Problem (2) is about the relative size of the spaces $\mathcal{R}(X)$ and $A(X)$. In view of Runge's theorem, we can replace $\mathcal{R}(X)$ by $\mathcal{H}(X)$, and so the problem concerns the approximation of functions with singularities on $\mathbb{C} \sim \text{int } X$ by functions with singularities on $\mathbb{C} \sim X$. Recalling that the approximation problem is local, we fix an open disc D, and look for ways of comparing the set of functions with singularities in $D \sim \text{int } X$ with the set of functions with singularities in $D \sim X$.

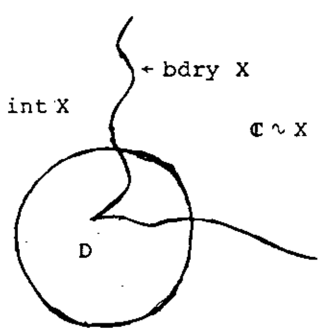


Figure 2

The key idea is to measure the relative sizes of these singularity sets by means of a capacity. A capacity is a nonegative increasing function

$$\gamma : 2^{\mathbb{C}} \to [0,\infty].$$

The capacity γ_B associated with B assigns a number $\gamma_B(E)$ to each set $E \subset \mathbb{C}$, and this number is a measure of the size of the collection of functions with singular support in E. The definition of $\gamma_B(E)$ for compact E is

$$\sup_f |a_1(f)|,$$

where f runs over all functions in the unit ball of $A(\mathbb{C} \sim E)$ (interpreted in the obvious way, since $\mathbb{C} \sim E$ is not compact), and $a_1(f)$ is the coefficient of $1/z$ in the expansion

$$f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

of f near ∞. For arbitrary sets E, $\gamma_B(E)$ is defined as

$$\sup\{\gamma_B(F) : F \subset E, F \text{ compact}\}.$$

It is not very surprising that if $R(X) = A(X)$, then $\gamma_B(D \sim X) = \gamma_B(D \sim \text{int } X)$ for every open disc D. The extraordinary thing is that the converse holds, so that a simple collection of numerical invariants characterizes rational approximation.

<u>Theorem</u> [24,9,1,10,11,19]. <u>Let</u> $B = L_p (2 \leq p < \infty)$, C, <u>or</u> lip $\alpha (0 < \alpha < 1)$. <u>Then</u> $R(X) = A(X)$ <u>if and only if</u> $\gamma_B(D \sim \text{int } X) = \gamma_B(D \sim X)$ <u>for all open discs</u> D.

(6.3) For practical purposes, the above theorem is of no use without a computational description of γ_B. In order to use the theorem to decide whether or not $R(X) = A(X)$ for a specific compact set X, we need to be able to compute $\gamma_B(E)$. The problem is markedly simplified by the fact that the capacity condition of the theorem is equivalent to the formally weaker condition:

There exists $\kappa > 0$ *such that* $\gamma_B(D \sim X) \geq \kappa\, \gamma_B(D \sim \text{int } X)$ *for each open disc* D.

This means that it is sufficient to identify γ_B up to multiplicative bounds, i.e. to find an explicitly-computable capacity $F : 2^{\mathbb{C}} \to [0,\infty]$ such that there exists a constant $\kappa > 0$ (which need not be known) such that

$$\kappa^{-1} F(E) \leq \gamma_B(E) \leq \kappa\, F(E)$$

for all sets $E \subset \mathbb{C}$.

(6.4) The case $B = \text{lip}\,\alpha (0 < \alpha < 1)$ is simplest to describe. In this case γ_B is comparable to $M_*^{1+\alpha}$, *lower* $(1+\alpha)$ – *dimensional Hausdorff content* [19]. This content is defined as follows.

For $h : [0,\infty] \to [0,\infty]$ and $E \subset \mathbb{C}$ we define

$$M_h(E) = \inf_{S} \sum_{D \in S} h(\text{diam } D)$$

where S runs over all countable coverings of E by open discs. For instance, in case $h(r) = r^\beta$, M_h is denoted M^β and is called β-dimensional Hausdorff content or size infinity approximating β-dimensional Hausdorff measure. We define

$$M_*^\beta(E) = \sup_h M_h(E)$$

where h runs over all functions $[0,\infty] \to [0,\infty]$ such that $h(r) \leq r^\beta$ and $r^{-\beta} h(r) \to 0$ as $r \downarrow 0$.

It turns out [20] that $M_*^\beta(E) = M^\beta(E)$ if E is open, so Theorem (6.2) specializes to the following explicit result.

Theorem. *Let* $B = \text{lip}\,\alpha (0 < \alpha < 1)$. *Then* $A(X) = R(X)$ *if and only if there exists* $\kappa > 0$ *such that*

$$M^{1+\alpha}(D \sim X) \geq \kappa\, M_*^{1+\alpha}(D \sim \operatorname{int} X)$$

for all open discs D.

Note that this is a metric condition. It implies that if $X = \phi(Y)$ where $\phi : \mathbb{C} \to \mathbb{C}$ is biLipschitzian, then $R(X) = A(X)$ if and only if $R(Y) = A(Y)$.

(6.5) In case $B = L^p$ $(2 \leq p < \infty)$, the capacity γ_B is of the potential theoretic kind, and has a couple of explicit descriptions [11]. One is that for compact E, $\gamma_B(E)$ is comparable to

$$\inf_u \|u\|_{W^{1,q}}$$

where u runs over all functions in $\mathcal{D}$ with $u \geq 1$ on E. Here q is the conjugate index to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, and the $W^{1,q}$ norm of u is the L^q norm of $|u| + |\nabla u|$. Another description is that $\gamma_B(E)$ is comparable to

$$\sup_\mu \mu(E)\,,$$

where μ runs over all positive measures supported on E such that the potential $K * \mu$ has L^p norm at most 1, where $K(z) = \frac{1}{|z|}$ is the Newtonian kernel. In the Hilbert space case, $p = 2$, which was the first to be cleared up [9], γ_B is the logarithmic capacity.

For all p, the capacity γ_B is a true Choquet capacity, and as a result it generates a corresponding fine topology on the plane. For those unfamiliar with such things, it may be helpful to describe a similar fine topology. The density topology on the plane is the topology for which a set N is a neighbourhood of a point a if $a \in N$ and $\mathbb{C} \sim N$ has area density 0 at a, i.e.

$$\lim_{r \downarrow 0} \frac{m\{B(a,r) \sim N\}}{\pi r^2} = 0\,.$$

This topology on $\mathbb{C}$ is finer than the Euclidean topology, yet it is still connected and Baire (i.e. second category in itself). The fine topology associated to γ_B is finer than the density topology. It may be described as follows. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be <u>finely-continuous</u> if given $\varepsilon > o$ there exists an open set U, with $\gamma_B(U) < \varepsilon$, such that f is continuous (in the usual Euclidean sense) on $\mathbb{C} \sim U$. The <u>fine topology generated by</u> γ_B is the least topology such that each finely-continuous function is continuous, i.e. it is the topology with subbase $\{f^{-1}(V) : f$ is finely-continuous, V is a Euclidean open set$\}$. In case $B = L^2$, this fine topology is the standard fine topology of potential theory, namely the pullback topology generated by the superharmonic functions.

In terms of the fine topology of γ_B, $B = L^p$, the solution of main problem (2) may be expressed as follows.

<u>Theorem</u>. <u>Let</u> $2 \leq p < \infty$, $B = L^p$, <u>and let</u> X <u>be compact in</u> $\mathbb{C}$. <u>Then</u> $R(X) = A(X)$ <u>if and only if the fine closure of</u> $\mathbb{C} \sim X$ <u>equals the set of fine accumulation points of</u> $\mathbb{C} \sim \text{int } X$.

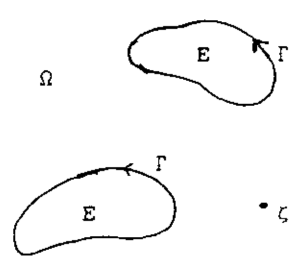
For $p = 2$, this is precisely the condition for each function, continuous on X and harmonic on int X, to be a <u>uniform</u> limit on X of functions harmonic on a neighbourhood of X. Thus L^2 analytic approximation is equivalent to L^∞ harmonic approximation. It would be interesting to see a more direct proof of this mysterious fact.

(6.6) The capacity γ_C corresponding to $B = C$ is the least well understood. It is known as <u>continuous analytic capacity</u>, and is usually denoted α. It was introduced by Dolženko. The associated outer capacity γ_C^*, defined by

$$\gamma_C^*(E) = \inf\{\gamma_C(U) : E \subset U \text{ open}\}$$

is the <u>analytic capacity</u> of Ahlfors. For any set E, $\gamma_c^*(E)$ is at most the logarithmic capacity of E, with equality for connected sets. For any set E, $\gamma_c(E)$ is at least the Newtonian capacity of E (defined using the kernel $|z|^{-1}$). In particular, $\gamma_c(E) \geq \{m(E)/\pi\}^{\frac{1}{2}}$. If $M^1(E) < \infty$, i.e. if E has finite outer length, then $\gamma_c(E) = 0$. If $M^\beta(E) > 0$ for some $\beta > 1$, then $\gamma_c(E) > 0$, so in terms of Hausdorff dimension the break-point for nullity of γ_c occurs at dimension 1. However, γ_c is not comparable to any M_h. A reference for the above facts if [6]. It is conjectured that γ_c^* is comparable to a one-dimensional Favard content (or integralgeometric content), calculable in terms of projections of E in almost every direction. See [12] for an account of progress on this, due to A. P. Calderon and S. Ya Havinson.

In principle, γ_c may be computed by a method due to P. Garabedian [22]. It suffices to calculate it for smoothly bounded compact sets with connected complement. Let E be a compact set with smooth boundary Γ, (possibly having several components) and with $\Omega = S^2 \sim E$ connected. Let $E^2(\Omega)$ denote the Smirnov space of all functions f, analytic in Ω, with non-tangential boundary values in $L^2(\Gamma, ds)$, where ds denotes arc length on Γ. Thus $E^2(\Omega)$ is a Hilbert space, with inner product



<u>Figure 3</u>

$$\langle f,g \rangle = \int_\Gamma f(z)\, \overline{g(z)}\, ds ,$$

and for $\zeta \in \Omega$, evaluation at ζ is a continuous linear functional on $E^2(\Omega)$. Thus there exists a function $z \to K(z,\zeta)$, belonging to

$E^2(\Omega)$, such that

$$f(\zeta) = \int_\Gamma f(z)\, \overline{K(z,\zeta)}\, ds$$

for all $f \in E^2(\Omega)$. The function $K(z,\zeta)$ is called the Szegö kernel. The formula of Garabedian is

$$K(\infty,\infty) = \frac{1}{2\pi\gamma_c(E)} .$$

The number $K(\infty,\infty)$ is the square of the norm of the functional $f \to f(\infty)$ on $E^2(\Omega)$. It can be computed to any desired degree of accuracy if enough terms of an orthonormal basis for $E^2(\Omega)$ are available. An orthonormal basis can be obtained by the Gram-Schmidt process from any independent set with dense span. If E has components $E_1, \dots, E_k$ and a_j is chosen in the interior of $E_j (j=1,\dots,k)$, then the set of functions $(z-a_j)^{-n}$ $(1 \le j \le k,\ n \ge 0)$ has dense span in $E^2(\Omega)$. Thus $\gamma_c(E)$ can in principle be computed to any desired accuracy. A lemma of E. Smith [22] shows that the following algorithm works. Let $u_0 = 1, u_1, u_2, \dots$ be an enumeration of the functions $(z-a_j)^{-n}$ $(1 \le j \le k,\ n \ge 0)$. Let $p_{ij} = \langle u_i, u_j \rangle$, and form the infinite matrix $P = (p_{ij})$ $(0 \le i < \infty,\ 0 \le j < \infty)$. Then $K(\infty,\infty)$ is the $(0,0)$ entry of P^{-1}. It is the limit of the numbers

$$\frac{\det(p_{ij} : 1 \le i, j \le n)}{\det(p_{ij} : 0 \le i, j \le n)} .$$

In practice, even if E is a simple kind of set with a small number of components, the number of computations involved in the algorithm is enormous, since the evaluation of each inner product involves a line integral. Thus it is prohibitively expensive to obtain more than two significant figures for $\gamma_c(E)$.

The main problem in the theory of uniform rational approximation is that of understanding γ_c. In particular, it is important to determine whether or not γ_c is <u>quasi-subadditive</u>, in the sense that

$$\gamma_C(E \cup F) \leq \kappa\{\gamma_C(E) + \gamma_C(F)\}$$

for some constant κ, independent of E and F. See [4].

§7. First main problem for the hairy class.

(7.1) The first main problem has been solved for $B = C$, and $B = \text{lip}\,\alpha (0 < \alpha < 1)$, but not for $B = L_p (2 \leq p < \infty)$.

For $\phi \in \mathcal{D}$, let $\text{spt}\phi$ denote the closed support of ϕ, the closure of $\{z \in \mathbb{C} : \phi(z) \neq 0\}$. Let $D(\phi)$ be the least closed disc containing $\text{spt}\phi$, and let $d(\phi)$ denote the diameter of $D(\phi)$. Let

$$\|\phi\|_* = \|\phi\|_\infty + d(\phi) \cdot \|\nabla\phi\|_\infty .$$

Theorem [24,21]. *Let* $B = C$ *or* $\text{lip}\,\alpha (0 < \alpha < 1)$. *Let* X *be a compact subset of* $\mathbb{C}$. *Let* $f \in B(\mathbb{C})$. *Then* $f \in R(X)$ *if and only if there exists* $\kappa > 0$ *such that*

$$\left| \int f\, \bar{\partial}\phi \, dm \right| \leq \kappa \, \|\phi\|_* \, \gamma_B\, (D(\phi) \sim X)$$

for all $\phi \in \mathcal{D}$.

The condition given in this theorem for $f \in R(X)$ may be viewed as a kind of weak analyticity condition. If $\text{spt}\,\phi \subset \text{int}\, X$, then it reduces to

$$\int f \, \bar{\partial}\phi \, dm = 0 ,$$

so it says that as a *distribution* f satisfies $\bar{\partial} f = 0$ on $\text{int}\, X$. It is well-known that this forces f to be analytic on $\text{int}\, X$. At a boundary point a, the integral condition places a restriction on f which is more or less stringent depending on how *thin* $\mathbb{C} \sim X$ is at the point a, where thinness is measured in terms of γ_B.

F. See [4].

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(7.2) It cannot be said that this rational approximation theory is i a satisfactory state. Not only are there many open problems, major and minor, but, more importantly, there is no coherent method for deriving the known results. Widely varying tools are used to tackle different classes B. It ought to be possible to develop axiomatic frameworks, one for smooth B and one for hairy B, within which a the central results can be proved. It is my hope that this article will prompt others to seek such frameworks.

The following list of references contains only the most recent sources available to me. Where possible, I referred to a textbook o monograph rather than the original paper.

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