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Theorems of Walsh–Lebesgue Type

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1 The Walsh–Lebesgue Theorem

This can be expressed as follows:

Let $X = \text{bdy}\ Y$, *where* $Y \subseteq \mathbb{C}$ *is compact and* $\mathbb{C} - Y$ *is connected. Then the (real-valued) harmonic polynomials are uniformly dense in* Re $C(X)$, *the space of real-valued continuous functions on* X.

If $f_1, \ldots, f_n$ are complex-valued continuous functions on a space X, let $P(f_1, \ldots f_n) = \mathbb{C}[f_1, \ldots, f_n]$ denote the algebra of all polynomials in $f_1, \ldots, f_n$, with complex coefficients, regarded as a subalgebra of $C(X)$. Then the conclusion of the Walsh–Lebesgue theorem can be stated in the form: the vector-space sum $P(z) + P(\bar{z})$ is uniformly dense in $C(X)$.

By a problem of "Walsh–Lebesgue type" I mean the following kind of thing. Let A_1 and A_2 be subalgebras of a Banach algebra A.

Describe the closure of $A_1 + A_2$. There are many such problems in analysis. I shall not attempt a general survey or a systematic analysis. I shall simply present a few results and examples on problems of this type.

First I shall discuss some approximation results on the circle. Then I shall give an honest generalization of the Walsh–Lebesgue theorem. Finally, I shall discuss approximation by real-valued functions.

2 Approximation on the Circle

Let S denote the unit circle, and D the open unit disk. Weierstrass proved that $P(z, \bar{z})$ is dense in $C(S)$. Nowadays we regard this result as a special case of Stone's theorem. But Stone's theorem is really a result about *real* algebras, in view of the hypothesis about complex conjugation. Dissatisfaction with this aspect of the matter led to much research, and one of the most striking results is *Wermer's maximality theorem* [6]:

Suppose $f \in C(S) \sim A(D)$. ($A(D)$ denotes the disk algebra, the algebra of all functions in $C(S)$ which extend continuously and analytically over D). *Then* $P(z, f)$ *is dense in* $C(S)$. In other words, the only property of $\bar{z}$ that matters in Weierstrass' result is the fact that it is not analytic.

The space $P(z, \bar{z})$ of trigonometric polynomials on S can also be written as $P(z) + P(\bar{z})$. This suggests another line of investigation: for which $f \in C(S)$ is $P(z) + P(f)$ dense in $C(S)$? This time, something more is required of f than non-analyticity. For instance, $P(z) + P(\bar{z}^2)$ is not dense in $C(S)$, since

$$\int_S g(z)dz = 0$$

whenever $g \in P(z) + P(\bar{z}^2)$. The general problem remains open, but we have the following:

THEOREM 2.1. *Let* $\psi : S \to S$ *be a homemorphism.*

(a) (F. and M. Riesz) *If* ψ *is singular, then* $P(z) + P(\psi)$ *is dense in* $C(S)$.

(b) (Browder and Wermer) *If* ψ *is orientation-reversing, then* $P(z) + P(\psi)$ *is dense in* $C(S)$ [1].

A singular homeomorphism is one which maps a set of full linear measure on S into a set of zero measure. There are many such maps. Condition (a) is a metric condition. Condition (b) is topological, and is therefore more appropriate and satisfying in the context of *uniform* approximation. It shows that, from this point of view, the crucial property of $\bar{z}$ for Weierstrass' theorem is that it reverses orientation. Note that in (b), z can be replaced by any orientation-preserving homeomorphism, so that the result concerns algebraic and topological categories, not analytic categories.

To prove the theorem, let μ be a measure on S which annihilates $P(z) + P(\psi)$. Since μ annihilates $P(z)$, the F. and M. Riesz theorem tells us that $\mu = h\mathrm{d}\theta$, where $h \in H^1_0$, the space of functions in the Hardy class H^1 which vanish at the origin. Defining the measure $\psi_{\#}\mu$ on S by the formula

$$\int f \mathrm{d}\, \psi_{\#}\mu = \int f \circ \psi \,\mathrm{d}\mu, \quad \text{for all } f \in C(S),$$

we have $\psi_{\#}\mu \perp P(z)$, hence $\psi_{\#}\mu = k\mathrm{d}\theta$, with $k \in H^1_0$. Assertion (a) of the theorem follows at once. To see (b), suppose ψ is orientation-reversing, and let

$$F(\mathrm{e}^{i\theta}) = \int_0^\theta k(\phi)\mathrm{d}\phi;$$

$$G(\mathrm{e}^{i\theta}) = \int_0^\theta h(\phi)\mathrm{d}\phi.$$

Then F, $G \in A(D)$, and $G \cdot \psi = F$ on S. An application of the argument principle now shows that $F(D) \subset F(S)$, so that if F is non-constant, then it maps S onto a set with non-empty interior. This is impossible, since F is absolutely-continuous. Thus $F = 0$, hence $\mu = 0$. The result follows.

We could ask which other self-adjoint Banach algebras A with spectrum S can replace $C(S)$ in Theorem 2.1 (b). It is easy to see that the algebra of functions with absolutely-convergent Fourier series will do. Garnett and I [2] obtained a partial positive result for the algebra $C^1(S)$, and we also showed that there is an example of a homeomorphism $\psi \in W^{1,1}$ (the algebra of absolutely-continuous functions on S) such that ψ is direction-reversing but $P(z) + P(\psi)$ is not dense in $W^{1,1}$. The positive result is as follows.

THEOREM 2.2. *Let* $\psi : S \to S$ *be a* $C^{1+\epsilon}$ *direction-reversing homeomorphism, with* $\psi \circ \psi = z$. *Then*

$$P(z) + P(\psi) \text{ is dense in } C^1(S).$$

Here $C^{1+\epsilon}$ means that the derivative $d\psi/d\theta$ satisfies a Hölder condition with exponent ϵ. This result is proved by "welding" the circle to produce a $C^{1+\epsilon/2}$ arc on the sphere, and reducing the assertion to be proved to a statement about removable singularities for H^1 classes with prescribed boundary behaviour. The example in $W^{1,1}$ is produced by reversing this procedure, and constructing an arc in the sphere with positive continuous analytic capacity, such that the harmonic measures for the two sides of the arc are mutually absolutely-continuous. The details are in [2].

3 Generalized Walsh–Lebesgue Theorem

The following result removes the harmonic functions from the Walsh–Lebesgue theorem, and leaves an algebraic and topological statement [5].

THEOREM 3.1. *Let* Φ and Ψ *be homeomorphisms of* $\mathbb{C}$ *on* $\mathbb{C}$ *with opposite orientations. Let* $X = \text{bdy } Y$, *where* $Y \subset \mathbb{C}$, Y *is compact, and* $\mathbb{C} \sim Y$ *is connected. Then*

$$P(\Phi) + P(\Psi) \text{ is dense in } C(X).$$

This also generalizes Theorem 2.1(b), in view of the Jordan–Schönflies theorem. The proof is not a new proof of the Walsh–Lebesgue theorem, because it uses both that theorem and 2.1(b). The first step is to use the generalized F. and M. Riesz theorem to localize to a single component U_n of int Y. Here it is important that harmonic measure for U_n is supported on the set of accessible boundary points (Fatou's theorem), which is a set with a *topological* description. The next step is to construct a direction-reversing homeomorphism of the circle, induced by the conformal maps to $\Phi(U_n)$ and $\Psi(U_n)$ and the map $\Psi \circ \Phi$. It is crucial that the prime ends of a simply-connected open set have a topological description. After that, it is a technical matter to reduce to the Browder–Wermer result.

One of the applications of Theorem 3.1 is the following [5].

THEOREM 3.2. *Let* Φ, Ψ, *and* Λ *be homeomorphisms of* $\mathbb{C}$ *onto* $\mathbb{C}$, *such that* Φ *and* Ψ *have opposite orientations. Then*

$$P(\phi, |\Lambda|^\alpha) + P(\Psi, |\Lambda|^\alpha)$$

is dense in $C(X)$ *for every compact* $X \subset \mathbb{C}$, and every $\alpha > 0$.

4 Real Functions

Now consider *real-valued* continuous functions on a compact Hausdorff space X. Let A_1 and A_2 be uniformly closed real subalgebras of $\operatorname{Re} C(X)$, containing the constants. Think, for example, of $X \subset \mathbb{R}^2$, $A_1 = \operatorname{clos} P(x)$, $A_2 = \operatorname{clos} P(y)$; or $X \subset \mathbb{R}^3$, $A_1 = \operatorname{clos} P(x, y)$, $A_2 = \operatorname{clos} P(z)$.

Let X_j be the compact quotient space of X obtained by identifying a and b whenever $f(a) = f(b)$ for all $f \in A_j$. Let $\Pi_j : X \to X_j$ be the projection. If $A_1 + A_2$ separates points on X, then $\Pi_1 \times \Pi_2 : X \to X_1 \times X_2$ is injective.

In [4], Marshall and I consider the problems: (1) When is $A_1 + A_2$ dense in $\operatorname{Re} C(X)$? (2) When is $A_1 + A_2$ closed? We also work on finitely-generated modules over A_1. We use abstract methods. Many people have used constructive methods on these problems, in the case of $P(x) + P(y)$, for instance Kolmogoroff, Havin, Havinson, Diliberto and Strauss, Ofman, Sprecher, Golomb, Mordashev, Ibragimov and Babaev. Buck has worked on the connection with functional equations, and has established some special cases of the following result, using abstract methods. There is also a thesis by Overdeck, which at this time we have not seen, which may well contain other cases.

A *trip* in X with respect to (A_1, A_2) is a finite ordered sequence $a_1, a_2, \ldots, a_n$ of points of X with $a_{j+1} \neq a_j$ and *either* $\Pi_1(a_2) = \Pi_1(a_1)$, $\Pi_2(a_3) = \Pi_2(a_2)$, $\Pi_1(a_4) = \Pi_1(a_3), \ldots,$ *or* $\Pi_2(a_2) = \Pi_2(a_1)$, $\Pi_1(a_3) = \Pi_1(a_2)$, $\Pi_2(a_4) = \Pi_2(a_3), \ldots$. A trip is a *round trip* if $a_1 = a_n$. Clearly, if $A_1 + A_2$ separates points, then a round trip must contain more than 2 points. The *orbit* of a point $a \in X$ is the set of all endpoints a_n of trips $a_1, \ldots, a_n$, with $a_1 = a_n$.

THEOREM 4.1. *Suppose each orbit is closed in X. Then $A_1 + A_2$ is dense in* Re $C(X)$ *if and only if there are no round trips.*

Proof. The map $f \to f \circ \pi_j$ carries Re $C(X_j)$ isometrically onto A_j, by Stone–Weierstrass, hence a real measure μ on X annihilates A_j if and only if the projected measure $\pi_{j\#}\mu$ on X_j is 0.

Suppose there are no round trips, and $A_1 + A_2$ is not dense. Let B denote the unit sphere of $A_1^\perp \cap A_2^\perp$. By Krein–Milman, there exists an extreme point $\mu \in B$. By the last paragraph, the restriction of μ to any Borel set which is a union of orbits is an annihilating measure. Let ν be the positive measure induced by the total variation measure $|\mu|$ on the (compact Hausdorff) space O of orbits. If ν is not a point mass, then we can write it as $\nu_1 + \nu_2$, where $\nu_j = \nu|O_j$ is non-zero, and $O = O_1 \cup O_2$ is a Borel decomposition. This induces a decomposition of μ as $\mu_1 + \mu_2$, where $\mu_j = \mu|E_j$ is non-zero, and E_j is a Borel union of orbits. Thus μ is not extreme. This contradiction shows that μ is supported on a single orbit E. Fix $a \in E$. Let $E_1 = \{a\}$, $E_2 = \pi_1^{-1}(E_1), E_3 = \pi_2^{-1}(E_2)$, $E_4 = \pi_1^{-1}(E_3)$, and so on. Each E_n is compact, and $E = \bigcup_n E_n$. There exists a first n with $|\mu|E_n > 0$. We may assume that $\mu_+ E_n > 0$, where $\mu = \mu_+ - \mu_-$ is the Haar decomposition. Choose $F_1 \subset E_n$ such that F_1 is Borel, F_1 supports $\mu_+|E_n$, and $\mu_-(F_1) = 0$. There exists $F_2 \subset E_{n+1} \sim E_n$ such that $\mu_-(F_2) = \mu_+(F_1)$ since otherwise there are round trips. There exists $F_3 \subset E_{n+2} \sim E_{n+1}$ such that $\mu_+(F_3) = \mu_-(F_2)$, and so on. The sets F_n are pairwise disjoint, since there are no round trips. Thus μ has infinite total variation. This contradiction establishes the theorem.

Havinson [3] has produced an ingenious example of a compact set $X \subset \mathbf{R}^2$ such that $P(x) + P(y)$ is not dense in Re $C(X)$, yet there are no round trips. Of course, the orbits are not closed, in fact, they are all dense.

Current research focuses on the case of dense orbits, and also on sums of more than two subalgebras.

5 Concluding Remarks

Problems of Walsh–Lebesgue type occur in a wide variety of areas.

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It might be worthwhile to develop the algebraic theory of the categories SA = sums of algebras and FSA = finite sums of algebras. Quite obviously, a sum of algebras is a module over the intersection an is contained in a unique enveloping algebra, and these properties allow the transfer of many properties of modules and algebras to this category.

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