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Uniform approximation by real functions

by

Donald E. Marshall and **Anthony G. O'Farrell** (Maynooth)

Abstract. Let $C(X)$ denote the space of real-valued continuous functions on a compact Hausdorff space X. We obtain a necessary and sufficient condition for the vectorspace sum A_1+A_2 of two subalgebras to be dense in $C(X)$. We solve the analogous problem for finitely-generated modules over a subalgebra of $C(X)$. Also, we determine the conditions under which these various spaces are closed.

In this paper, we consider some questions about approximation by real functions. The Stone-Weierstrass theorem completed the theory of qualitative uniform approximation by elements of an algebra of real functions. We study two related structures, namely modules and finite sums of algebras. The reader will observe that the nature of the subject is essentially geometric, in contrast to the topological and metric character [3], [6] of complex polynomial approximation. Even a simple rotation of a set in R^n can radically alter the closure of certain polynomial spaces.

Let us introduce some notation. If X is a compact Hausdorff space, then $C(X)$ denotes the space of all continuous, real-valued functions on X. If A is a subset of $C(X)$, then $\overline{A}$ denotes the closure of A with respect to the uniform norm, $\|\cdot\|$, on X. If $f_1, f_2, \ldots, f_n$ belong to $C(X)$, then $P(f_1, f_2, \ldots, f_n)$ denotes the algebra of all polynomials in $f_1, f_2, \ldots, f_n$ with real coefficients.

Our main result is Proposition 2, which gives a necessary and sufficient condition for the sum of two subalgebras of $C(X)$ to be dense in $C(X)$.

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1. Sums of algebras.

(1.1) We begin with a special case. For which functions f and g, belonging to $C(X)$, is

$$\overline{P(f)+P(g)} = C(X)? \tag{1}$$

Clearly it is necessary that the map from X to R^2, given by $F(x) = (f(x), g(x))$, be injective. Suppose F is injective. If Y is the image of X under F, then Y is homeomorphic to X. A moment's thought reveals that (1) holds if and only if

$$\overline{P(x)+P(y)} = C(Y), \tag{2}$$

where x and y denote the coordinate functions in $\mathbf{R}^2$. So our problem is to characterize the compact sets Y in $\mathbf{R}^2$ for which (2) holds.

To analyze this question, we introduce the concept of a *trip* in Y. A *trip* in Y is a finite ordered subset $\{a_1, \ldots, a_n\}$ of Y with $a_i \neq a_{i+1}$ $(i = 1, \ldots, n-1)$, and either $a_1 = (x_1, y_1)$, $a_2 = (x_1, y_2)$, $a_3 = (x_2, y_2)$, $a_4 = (x_2, y_3)$, ..., or $a_1 = (x_1, y_1)$, $a_2 = (x_2, y_1)$, $a_3 = (x_2, y_2)$, ... A trip with at least two distinct points is called a *round trip* if $a_1 = a_n$. The relation on Y, defined by setting $a \approx b$ if a and b belong to some trip in Y, is an equivalence relation. The equivalence classes we call *orbits*.

PROPOSITION 1. *Let Y be a compact subset of $\mathbf{R}^2$ with all orbits closed. Then $P(x)+P(y)$ is uniformly dense in $C(Y)$ if and only if Y contains no round trip.*

Proof. The necessity of the condition is clear. For if Y contains a round trip, then it contains a round trip with an even number of distinct points, say $\{a_1, a_2, \ldots, a_{2n}, a_1\}$. The alternating sum

$$f(a_1)-f(a_2)+f(a_3)-\ldots-f(a_{2n})$$

vanishes for all f in $P(x)+P(y)$. Thus $P(x)+P(y)$ is not dense in $C(Y)$.

Conversely, suppose Y contains no round trip. We will prove $P(x)+P(y)$ is dense in $C(Y)$ by showing that the only annihilating measure for $P(x)+P(y)$ is the zero measure. Let $M(Y)$ denote the space of all real, finite, Borel-regular measures on Y. Let π_1 and π_2 denote the orthogonal projections of Y into the coordinate axes. That is, $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$, whenever (x, y) belongs to Y.

Suppose, contrary to our assertion, there exist nonzero annihilating measures on Y for $P(x)+P(y)$. Let K denote the closed unit ball of the space of annihilating measures. That is,

$$K = M(Y) \cap P(x)^{\perp} \cap P(y)^{\perp} \cap \{\mu\colon \|\mu\| \leqslant 1\}.$$

Then K is weak-* compact and convex. By the Krein–Milman theorem, there exists an extreme measure μ in K. The support of μ must be contained in a single orbit. To see this, note that a measure ν annihilates $P(x)$ if and only if $\nu(\pi_1^{-1}(E)) = 0$ for every Borel subset E of $\mathbf{R}$, by the Stone–Weierstrass theorem. If a Borel set C is a union of orbits, then $\pi_1^{-1} \circ \pi_1(a)$ is contained in C, whenever a belongs to C. Thus the restriction $\mu|C$ of μ to C annihilates $P(x)$. Similarly, $\mu|C$ annihilates $P(y)$. Since μ is extreme, it must be supported on a single orbit. Call the orbit C.

Fix a point a_1 in C. Then C may be written as the union of compact sets C_i, where $C_1 = \{a_1\}$, $C_2 = \pi_1^{-1} \circ \pi_1(C_1)$, $C_3 = \pi_2^{-1} \circ \pi_2(C_2)$, $C_4 = \pi_1^{-1} \circ \pi_1(C_3)$, and so on. Clearly we have $C_i \subset C_{i+1}$ for each i. Thus there exists a positive integer n, such that $|\mu|(C_{2n})>0$, where $|\mu|$ denotes the total variation measure of μ. Since μ annihilates $P(x)$, it follows from the Stone-Weierstrass theorem, again, that $\mu(C_{2n}) = 0$. Hence $\mu_+(C_{2n}) = \mu_-(C_{2n})>0$, where $\mu = \mu_+ - \mu_-$ is the Haar decomposition of μ. Choose a Borel subset E_0 of C_{2n}, such that $\mu_-(E_0) = 0$ and $\mu_+(E_0)>0$. Since μ annihilates $P(y)$, it follows that $\mu(\pi_2^{-1} \circ \pi_2(E_0)) = 0$, so we may choose a Borel set E_1, such that $E_1 \subset \pi_2^{-1} \circ \pi_2(E_0) \subset C_{2n+1}$, $E_1 \cap E_0 = \emptyset$, $\mu_+(E_1) = 0$, and $\mu_-(E_1) \geqslant \mu_+(E_0)$. Continuing this process, choose a Borel set E_2, such that

$E_2 \subset \pi_1^{-1} \circ \pi_1(E_1) \subset C_{2n+2}$, $E_2 \cap E_1 = \emptyset$, $\mu_-(E_2) = 0$, and $\mu_+(E_2) \geqslant \mu_-(E_1)$, and so on.

We see that the resulting sets, $E_0, E_1, E_2, \ldots$, are pairwise disjoint. For otherwise, there would exist positive integers k and m, with $k<m$, and a trip $\{b_k, b_{k+1}, \ldots, b_m\}$, such that b_i belongs to E_i for $i = k, \ldots, m$, and b_m belongs to $E_m \cap E_k$. But then there would exist trips $\{a_1, a_2, \ldots, a_{k-1}, b_k\}$ and

$$\{a_1, a_2', a_3', \ldots, a_{k-1}', b_m\}$$

with a_i and a_i' in C_i, for $i = 2, \ldots, k-1$. Thus, the set

$$\{a_1, a_2, \ldots, a_{k-1}, b_k, b_{k+1}, \ldots, b_m, a_{k-1}', \ldots, a_2', a_1\}$$

would contain a round trip. This would contradict our assumption on Y.

Since the E_i are pairwise disjoint, and $|\mu|(E_i) \geqslant \mu_+(E_0) > 0$ for each i, it follows that the total variation of μ is infinite. This contradiction establishes the proposition.

(1.2) Now we turn to the general situation. Let X be a compact Hausdorff space. Let A_1 and A_2 be any two subalgebras of $C(X)$ that contain the constants. The problem is to decide when $A_1 + A_2$ is dense in $C(X)$.

For $i = 1, 2$, let X_i be the quotient space of X obtained by identifying the points a and b whenever $f(a) = f(b)$ for each f in A_i. Let π_i be the natural projection of X onto X_i. We define a *trip with respect to* (A_1, A_2) as a finite ordered set $\{a_1, \ldots, a_n\}$, contained in X, such that $a_i \neq a_{i+1}$ for $i = 1, 2, \ldots, n-1$, and either $\pi_1(a_2) = \pi_1(a_1)$, $\pi_2(a_3) = \pi_2(a_2)$, $\pi_1(a_4) = \pi_1(a_3), \ldots$, or $\pi_2(a_2) = \pi_2(a_1)$, $\pi_1(a_3) = \pi_1(a_2)$, $\pi_2(a_4) = \pi_2(a_3), \ldots$ As before, we say that a trip is a *round trip* if $n > 1$ and $a_1 = a_n$. Notice that X contains a two-point round trip $\{a_1, a_2, a_1\}$ with respect to (A_1, A_2) if and only if $A_1 + A_2$ fails to separate points on X. The relation, defined by setting $a \approx b$ if a and b belong to some trip with respect to (A_1, A_2), is an equivalence relation on X. We call the equivalence classes the *orbits* of X with respect to (A_1, A_2).

PROPOSITION 2. *Let X be a compact Hausdorff space. Let A_1 and A_2 be subalgebras of $C(X)$ that contain the constants. Suppose all orbits are closed. Then $A_1 + A_2$ is uniformly dense in $C(X)$ if and only if X contains no round trip with respect to (A_1, A_2).*

The proof of Proposition 2 is similar to the proof of Proposition 1. The special case already involves all the essential difficulties.

2. Applications and examples.

(2.1) Here are some sets Y, contained in $\mathbf{R}^2$, for which (2) fails.

(a) $\{(0,0), (1,0), (1,1), (0,1)\}$.

(b) $\{(0,0), (1,0), (1,2), (2,2), (2,1), (0,1)\}$.

(c) $\{(0,0), (1,0), (1,2), (3,2), (3,3), (2,3), (2,1), (0,1)\}$.

(d) Any set Y with positive area, since any such set contains the vertices of some rectangle with sides parallel to the axes.

(2.2) Let Y be the union of two parallel line segments in $\boldsymbol{R}^2$, not parallel to either axis. Then (2) holds, as is easily seen from Proposition 1. On the other hand, if Y consists of three sufficiently long parallel line segments, then a little geometry shows that Y contains a four-point or a six-point round trip with respect to $(P(x), P(y))$, hence (2) fails.

PROBLEM 1. *Let Y be a compact subset of $\boldsymbol{R}^2$ with empty interior. Do there exist functions f and g in $C(Y)$ such that $P(f)+P(g)$ is dense in $C(Y)$?*

If Y is totally disconnected, then there is an injective function f in $C(Y)$. Thus $P(f)$ is dense in $C(Y)$. The general situation appears difficult.

(2.3) Let S denote the unit circle, and define

$$l_\theta(x, y) = x\cos\theta + y\sin\theta$$

whenever $0 \leqslant \theta \leqslant 2\pi$, $(x, y) \in S$. Then for each positive integer n, the sum

$$\mathop{+}_{j=1}^{2n} P(l_{j\pi/n}) \tag{3}$$

fails to be dense in $C(S)$. An annihilating measure for the space (3) is provided by the alternating sum of of point masses:

$$\sum_{k=1}^{2n} (-1)^k \left(\cos\frac{k\pi}{n}, \sin\frac{k\pi}{n}\right).$$

(2.4) *A necessary and sufficient condition that $P(l_\theta)+P(l_\varphi)$ be dense in $C(S)$ is that $\theta-\varphi$ be an irrational multiple of π.*

The necessity follows from (2.3). To see the sufficiency, suppose $(\theta-\varphi)/\pi$ is irrational. Let $R(\theta) \in \mathbf{O}(2)$ (the group of isometries of $\boldsymbol{R}^2$, cf. [2]) denote reflection in the line

$$x\sin\theta - y\cos\theta = 0.$$

Let G denote the subgroup of $\mathbf{O}(2)$ generated by $R(\theta)$ and $R(\varphi)$. Then the orbits of S with respect to $(P(l_\theta), P(l_\varphi))$ are precisely the orbits, in the usual sense, of S under the action of G. Since $(\theta-\varphi)/\pi$ is irrational, the orbits of G are all dense in S, and we obtain the result by an ergodicity argument (cf. (2.6) below).

Note that the orbits are not closed, so we cannot apply Proposition 2.

(2.5) Let $X \subset \boldsymbol{R}^2$ be compact, and let x, y and z denote the coordinate functions. When is the sum $P(x, y)+P(y, z)+P(z, x)$ dense in $C(X)$? The annihilating measures for this sum of algebras assign zero measure to each union of lines parallel to any coordinate axis. So we may imitate the technique of Proposition 1, and answer the question in terms of suitable round trips. This method does not seem to work for $P(x)+P(y)+P(z)$. Notice that if

$$X = \{(0, 0, 1), (0, 1, 1), (1, 0, 0), (2, 1, 0), (1, 2, 1), (2, 2, 1)\},$$

then $P(x, y)+P(z)$, $P(x, z)+P(y)$, and $P(y, z)+P(x)$ are all dense in $C(X)$, but $P(x)+P(y)+P(z)$ is not.

PROBLEM 2. *Let X be a compact Hausdorff space, and let $A_1, A_2, \ldots, A_n$ be subalgebras of $C(X)$ that contain the constants. Give conditions that are necessary and sufficient for $A_1+\ldots+A_n$ to be dense in $C(X)$.*

(2.6) Let S^2 be the unit sphere in $\boldsymbol{R}^3$, and let u, v and w be three real-valued linear functions on $\boldsymbol{R}^3$. Suppose the gradients ∇u, ∇v and ∇w are linearly independent, and that at least two of the angles that the vectors $\nabla u \times \nabla v$, $\nabla v \times \nabla w$, $\nabla w \times \nabla u$ make with one another are irrational multiples of π. Then $P(u, v)+P(v, w)+P(w, u)$ is dense in $C(S^2)$.

To prove this, suppose μ is a measure on S^2 that annihilates $P(u, v)+P(v, w)+$ $+P(w, u)$. Let $R \in \mathbf{O}(3)$ [2] denote reflection in the plane through the origin spanned by ∇u and ∇v. Let S and $T \in \mathbf{O}(3)$ be similarly defined, with (u, v) replaced by (v, w) and (w, u) respectively. Then for each Borel set $E \subset S^2$, we have

(4) $$\mu(R(E)) = \mu(S(E)) = \mu(T(E)) = -\mu(E).$$

Let G denote the subgroup of $\mathbf{SO}(3)$ [2] generated by RS and ST, and let H be the connected component of the identity in the closure of G in $\mathbf{SO(3)}$. Then H is a sub-Lie group of $\mathbf{SO}(3)$. Since $H \neq \{1\}$, it equals $\mathbf{SO}(3)$ unless it consists of the rotations about some fixed axis [4]. Since at least two of the angles between $\nabla u \times \nabla v$, $\nabla v \times \nabla w$, and $\nabla w \times \nabla u$ are irrational multiples of π, it follows that H contains all the rotations about two distinct axes, and so H equals all $\mathbf{SO}(3)$.

By (4), the measure μ is invariant under the action of G, and hence under the action of $\mathbf{SO}(3)$. Thus μ is a multiple of the invariant measure on S^2, as a homogeneous space under the action of $\mathbf{SO}(3)$, and hence μ is just a multiple of surface area on S^2. But this contradicts (4), unless $\mu = 0$.

3. Modules.

(3.1) Let X be a compact Hausdorff space, and let A be a subalgebra of $C(X)$ that contains the constants. Let X_A be the quotient space induced by A, and let π_A be the natural projection of X onto X_A. Suppose B is a subspace of $C(X)$ that is an A-module, i.e. $AB \subset B$. Then, by modifying an argument of de Branges [1, 5], we see that the extreme norm 1 annihilating measures for B are supported on the fibres of π_A. We deduce the following:

PROPOSITION 3. *B is dense in $C(X)$ if and only if the restriction of B to each fibre $\pi_A^{-1}(y)$ is dense in $C(\pi_A^{-1}(y))$.*

Suppose B is a finitely-generated A-module. That is, $B = f_1 A+\ldots+f_n A$, where the f_j are in $C(X)$. Then *B is dense in $C(X)$ if and only if for each $y \in X_A$ the matrix*

$$\begin{bmatrix} f_1(x_1) & \ldots & f_n(x_1) \\ \ldots & \ldots & \ldots \\ f_1(x_m) & \ldots & f_n(x_m) \end{bmatrix}$$

has rank at least m, whenever $x_1, \ldots, x_m$ are distinct points in $\pi_A^{-1}(y)$.

For example, let X be a compact subset of $\boldsymbol{R}^2$, and let n be a non-negative integer. Then $P(x)+yP(x)+\ldots+y^nP(x)$ is dense in $C(X)$ if and only if each line parallel to the y-axis meets X in at most $n+1$ points. This is interesting because the space

$$\overline{P(x)}+y\,\overline{P(x)}+\ldots+y^n\,\overline{P(x)}$$

is not usually closed. For instance, if $X = \{(x, y)\colon 0 \leqslant x \leqslant 1,$ and $y = 0$ or $y = x\}$, then $P(x)+yP(x)$ is dense in $C(X)$, but the continuous function $y^{\frac{1}{2}}$ does not belong to $(\overline{P(x)}+y\overline{P(x)})$.

4. Closed sums.

(4.1) Let X be a compact Hausdorff space. Let A_1 and A_2 be closed subalgebras of $C(X)$. In some cases, it turns out that A_1+A_2 is a closed subspace of $C(X)$. For instance, if Y consists of two parallel line segments in $\boldsymbol{R}^2$, one can check that $\overline{P(x)}+\overline{P(y)}$ equals either $\overline{P(x)}$, $\overline{P(y)}$, or $C(Y)$. On the other hand, there exist sets $Y\subset\boldsymbol{R}^2$ such that $P(x)+P(y)$ is dense in $C(Y)$, but $\overline{P(x)}+\overline{P(y)}$ is not closed. For example, let $Y = \{(0,0), (-1,1), (-1,-\frac{1}{2}), (\frac{1}{2},-\frac{1}{2}), (\frac{1}{2},\frac{1}{4}), (-\frac{1}{4},\frac{1}{4}), (-\frac{1}{4},-\frac{1}{8}), (\frac{1}{8},-\frac{1}{8}), \ldots\} = \{a_0, a_1, a_2, a_3, \ldots\}$, say. Let h be the continuous function on Y defined by setting $h(a_0) = 0$, $h(a_n) = (-1)^n/n$ $(n = 1, 2, \ldots)$. Then it is easy to see that there cannot exist continuous functions f and g on $\boldsymbol{R}$, such that $f(x)+g(y) = h(x, y)$ for all $(x, y) \in Y$.

(4.2) We propose to characterize the spaces X, and the closed subalgebras A_1 and A_2 of $C(X)$, for which A_1+A_2 is closed in $C(X)$.

Let A be a subalgebra of $C(X)$ that contains the constants. Let X_A and π_A be the associated quotient space and projection. For $f\in C(X)$, let $d(f, A)$ be the distance from f to the algebra A, and for $Y\subset X$ let $\operatorname{var}_Y f$ be the variation of f on the set Y. That is,

$$d(f, A) = \inf_{g\in A}\sup_{x\in X}|f(x)-g(x)|\,,$$

$$\operatorname{var}_Y f = \sup_{x,y\in Y} f(x)-f(y)\,.$$

The following lemma is due to A. Pełczyński [5, p. 50]. We include a proof for the reader's convenience.

LEMMA 1. *Suppose $f\in C(X)$. Then*

$$d(f, A) = \tfrac{1}{2}\sup_{Y\in X_A}\ \operatorname*{var}_{\pi_A^{-1}(y)} f\,.$$

Proof. Fix $f\in C(X)$. Clearly, $d(f, A)$ is no smaller than the right-hand side. To prove the opposite inequality, let K be the set of norm 1 measures μ, belonging to $M(X)$, orthogonal to A and such that $\int f\,d\mu = d(f, A)$. By the Hahn–Banach theorem, K is nonempty. Let μ be an extreme point of K. Suppose there exist Borel sets E and F, contained in X_A, such that $X_A = E \cup F$, $E \cap F = \emptyset$, $|\mu|(\pi_A^{-1}(E))>0$, and $|\mu|(\pi_A^{-1}(F))>0$. Let λ be the restriction of μ to $\pi_A^{-1}(E)$, and ν be the restriction of μ to $\pi_A^{-1}(F)$. Then λ and ν annihilate A, and

$$\begin{aligned} d(f, A) &= \int f\,d\nu+\int f\,d\lambda \\ &= \|\nu\| \int f\frac{d\nu}{\|\nu\|}+\|\lambda\| \int f\frac{d\lambda}{\|\lambda\|} \\ &\leqslant \|\nu\| d(f, A)+\|\lambda\| d(f, A) \\ &= d(f, A). \end{aligned}$$

Thus

$$\int f\frac{d\lambda}{\|\lambda\|} = \int f\frac{d\nu}{\|\nu\|} = d(f, A),$$

hence $\lambda/\|\lambda\|$ and $\nu/\|\nu\|$ belong to K. Since

$$\mu = \|\lambda\|\left(\frac{\lambda}{\|\lambda\|}\right)+\|\nu\|\left(\frac{\nu}{\|\nu\|}\right),$$

μ cannot be extreme. This contradition shows that μ is supported on some fibre $\pi_A^{-1}(y)$. Now,

$$\int f\,d\mu = \int (f-\alpha)\,d\mu$$

for all $\alpha \in R$. Hence

$$\left|\int f\,d\mu\right|\leqslant \tfrac{1}{2} \operatorname*{var}_{\pi_A^{-1}(y)} f.$$

This proves the lemma.

PROPOSITION 4. *Let A_1 and A_2 be closed subalgebras of $C(X)$ that contain the constants. Let (X_1, π_1), (X_2, π_2), and (X_{12}, π_{12}) be the quotient spaces and projections associated with the algebras A_1, A_2, and $A_1 \cap A_2$, respectively. Then A_1+A_2 is closed in $C(X)$ if and only if there exists a positive real number c such that*

$$\sup_{z\in X_{12}} \operatorname*{var}_{\pi_{12}^{-1}(z)} f\leqslant c \sup_{y\in X_2} \operatorname*{var}_{\pi_2^{-1}(y)} f$$

for all f in A_1.

Proof. The linear isomorphism

$$\frac{A_1}{A_1 \cap A_2} \to \frac{A_1+A_2}{A_2} \subset \frac{C(X)}{A_2}$$

is continuous. By the open mapping theorem, there exists $c>0$ such that $d(f, A_1 \cap A_2)\leqslant c\,d(f, A_2)$ holds for all f in A_1, if and only if $(A_1+A_2)/A_1$ is closed, and this happens if and only if A_1+A_2 is closed. The proposition now follows from Lemma 1.

(4.3) For example, if Y is the closure of the interior of any ellipse in R^2, then

$\overline{P(x)}+\overline{P(y)}$ is closed in $C(Y)$. However, if Y is the closure of the region in R^2 bounded by the lines $2y = x$, $y = x$, and $y = 1$, then $\overline{P(x)+P(y)}$ is not closed in $C(Y)$.

(4.4) We can also answer the analogous question for finitely-generated modules which contain the constants. An elementary application of the open mapping theorem gives the following lemma.

LEMMA 2. *Let D be a Banach space. Let $B_1, B_2, \ldots, B_n$ be closed subspaces of D. Then $B = B_1+\ldots+B_n$ is closed in D if and only if there exists $K<\infty$ such that each b in B has a representation*

$$b = b_1+\ldots+b_n,$$

where $b_i \in B_i$ for $i = 1, 2, \ldots, n$, and

$$\max_{1\leq i\leq n} \|b_i\| \leq K\|b\|.$$

Now suppose A is a closed subalgebra of $C(X)$ that contains the constants, and suppose B is an A-module of the form: $B = A+f_1A+\ldots+f_nA$, where $f_i \in C(X)$ for $i = 1, \ldots, n$. Then *B is closed in $C(X)$ if and only if there exists $K<\infty$ such that each b in B has a representation of the form $b = a_0+f_1a_1+\ldots+f_na_n$, where $a_i \in A$, for $i = 1, \ldots, n$, and $\max_{0\leq i\leq n} \|a_i\| \leq K\|b\|$.*

To see this, choose a positive number c, so large that f_i+c is invertible in $C(X)$ for $i = 1, 2, \ldots, n$. Then $(f_i+c)A$ is closed. Clearly, $B = A+(f_i+c)A+\ldots+ +(f_n+c)A$. By Lemma 2, B is closed if and only if there exists a constant $K<\infty$, such that each b in B has the form $b = a_0+\sum_{i=1}^{n}(f_i+c)a_i$, where

$$\max\{\|a_0\|, \|(f_1+c)a_1\|, \ldots, \|(f_n+c)a_n\|\} \leq K\|b\|.$$

Clearly, this occurs if and only if there exists a constant $K' \leq \infty$ such that

$$\max_{0\leq i\leq n} \|a_i\| \leq K'\|b\|.$$

The proposition follows.

For example, let Y be a compact set in R^2 such that every line parallel to the y-axis meets Y in at least $n+1$ distinct points or not at all. For each $x \in \pi_1(Y)$, let $\delta(x)$ denote the supremum of the positive numbers η for which there exist $n+1$ distinct points $a_1, \ldots, a_{n+1}$ in $\pi_1^{-1}(x)$ such that $|a_i - a_j| > \eta$ whenever $i \neq j$. Then the module

$$\overline{P(x)}+y\,\overline{P(x)}+\ldots+y^n\,\overline{P(x)}$$

is closed in $C(Y)$ whenever

$$\inf\{\delta(x) : x \in \pi_1(Y)\} > 0.$$

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Added in proof. S. Ya. Havinson (A Chebyshev theorem for approximation of a function of two variables by sums of the type $\varphi(x)+\psi(y)$, Math USSR Izvestia 3 (1969), pp. 617–632, especially pp. 620–622) constructed an example which shows that Proposition 1 fails without the assumption of closed orbits. He also gave an example of the phenomenon exhibited in (4.1).

R.C. Buck and J. Overdeck were aware of special cases of Proposition 1.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES

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