# A REGULAR UNIFORM ALGEBRA WITH A CONTINUOUS POINT DERIVATION OF INFINITE ORDER

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## 1. *Introduction*

Let $A$ be a semisimple commutative Banach algebra with identity. Then $A$ is an algebra of continuous functions on its maximal ideal space $X$ [**1, 5**]. The algebra $A$ is *regular* if, given a (weak-star) closed set $E \subset X$ and a point $a \in X \sim E$, there exists a function $f \in A$ such that $f = 0$ on $E$, and $f(a) \neq 0$; equivalently, the hull-kernel topology on $X$ coincides with the weak-star topology [**1**; p. 56]. A *point derivation* [**4**] of order $N$ (respectively, $\infty$) at a point $a \in X$ is a sequence $d_0, d_1, \ldots$ of linear functions from $A$ to $\mathbb{C}$ such that

$$d_0 f = f(a)$$

$$d_n(fg) = \sum_{k=0}^{n} (d_k f)(d_{n-k} g)$$

holds for all $f, g \in A$ and $n = 1, 2, \ldots, N$ (respectively, $n = 1, 2, \ldots$). A point derivation is *continuous* if $d_n$ is continuous for $n \leqslant N$ (respectively, all $n$). The algebra $A$ is a *uniform* algebra if the norm on $A$ is the supremum norm on $X$.

Looking at simple examples, one might suppose that for uniform algebras the existence of nontrivial point derivations is inconsistent with regularity. This is supported by the simple folk theorem that if $A$ is a *uniformly normal* uniform algebra on $X$, then $A = C(X)$, the algebra of all continuous complex-valued functions on $X$; there are no nontrivial first order point derivations on $C(X)$. Here, "uniformly normal" means that there exists $M > 0$ such that, given disjoint closed sets $E, F \subset X$, there exists $f \in A$ with $\|f\| \leqslant M$ such that $f = 0$ on $E$ and $f = 1$ on $F$. McKissick [**7**] has shown that there are regular uniform algebras other than $C(X)$.

In the present note we show how to construct a regular uniform algebra which admits a nontrivial continuous point derivation of infinite order. The construction combines ideas from McKissick's paper [7] and from [**9**].

## 2. *Swiss cheeses*

Let $X$ be a compact subset of the complex plane. Then $R(X)$ denotes the uniform closure on $X$ of the algebra $R_0(X)$ of rational functions with poles off $X$. Let $a \in X$, and define

$$d_n f = \frac{f^{(n)}(a)}{n!}$$

for $f \in R_0(X)$ and $n = 0, 1, 2, \ldots$. If all these functions $d_n$ extend continuously to $R(X)$, then they are the terms of a continuous point derivation of infinite order. Let $A_m(a)$ denote the annulus

$$\{z \in \mathbb{C} : 2^{-m-1} < |z-a| < 2^{-m}\}$$

Received 9 May, 1978.

[BULL. LONDON MATH. SOC., 11 (1979), 41–44]

and let $\gamma$ denote (inner) analytic capacity. Then Hallstrom's Theorem **[6]** states that $d_n$ is continuous if and only if

$$\sum_{m=1}^{\infty} 2^{(n+1)m} \gamma(A_m(a) \sim X) < \infty.$$

Thus, a sufficient condition for *all* the $d_n$ to be continuous is that

$$\sum_{m=1}^{\infty} m^m \gamma(A_m(a) \sim X) < \infty. \tag{1}$$

A *Swiss cheese* is a compact set obtained by deleting from the closed unit disc a dense open subset, the union of a countable collection of open discs $D_j$, with radii $r_j$, such that $\sum_1^{\infty} r_j < \infty$. If we suppose that each excised disc meets at most three annuli $A_n(0)$, then condition (1) for $a = 0$ follows from

$$\sum_{m=1}^{\infty} m^m R_m < \infty, \tag{2}$$

where $R_m$ is the sum of the radii of those discs which meet $A_m(0)$. This is so, because the analytic capacity of a union of open discs is at most the sum of the radii. There is also an elementary argument on the lines of [**9**; Lemma 2], which proves that condition (2) is sufficient for all $d_n$ to be continuous, for $a = 0$.

We need the following [**7**; p. 393, Lemma 2; **10**; (27.6)]

McKissick's Lemma. *Let $D$ be an open disc in the plane, with centre $a$ and radius $r > 0$, and let $\varepsilon > 0$ be given. Then there exists a sequence $\{D_j\}_1^{\infty}$ of open discs and a sequence $\{f_j\}_1^{\infty}$ of rational functions such that*

(1) $D_j \subset \{z : r/2 < |z-a| < r\}$,

(2) $\sum_{j=1}^{\infty}$ radius $(D_j) < \varepsilon$,

(3) *the poles of $f_j$ lie in* $\bigcup_{i=1}^{j} D_i$,

(4) *$\{f_j\}$ converges uniformly on $C \sim \bigcup_{j=1}^{\infty} D_j$ to a function which is identically zero off $D$ and nowhere zero on $D \sim \bigcup_{j=1}^{\infty} D_j$.*

Condition (1) is not part of the result as given by McKissick, but there is no difficulty in modifying his construction to ensure (1) holds.

## 3. *The construction*

We abbreviate $A_m(0)$ by $A_m$.

Let $\{(a_n, r_n)\}_1^{\infty}$ be an enumeration of all the pairs $(a, r)$, consisting of a rational point $a \in \mathbb{C}$ and a rational radius $r > 0$ such that one of the following holds:

(A) $0 < |a| \leqslant 1$, $r < |a|/2$, and $r < 1-|a|$.

(B) $a = 0$, and $r = 2^{-m}$ for some positive integer $m$.

Let $D_n$ be the open disc with centre $a_n$ and radius $r_n$.

If $(a_n, r_n)$ is of type (A), then $D_n$ meets at least one and at most three of the annuli $A_m$.

Let $m$ be the largest integer with $D_n \cap A_m \neq \varnothing$. Let $\varepsilon_n = 2^{-m-n} m^{-m}$, and apply McKissick's Lemma with $(D, \varepsilon) = (D_n, \varepsilon_n)$. Let $D_{nj}$ $(j = 1, 2, \ldots)$ be the resulting open discs, and let $r_{nj} = \text{radius } (D_{nj})$, so that

$$\sum_{j=1}^{\infty} r_{nj} < \varepsilon_n. \tag{3}$$

If $(a_n, r_n)$ is of type (B), with $r_n = 2^{-m}$, then let $\varepsilon_n = 2^{-m-n} m^{-m}$, and apply McKissick's Lemma to $(D_n, \varepsilon_n)$. By condition (1) of that lemma, the resulting open discs $D_{nj}$ all lie in the annulus $A_m$. Their radii, $r_{nj}$ satisfy condition (3).

Let $X$ be the Swiss cheese obtained by deleting

$$\bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} D_{nj}$$

from the closed unit disc. Then it is clear that this cheese satisfies condition (2) (indeed, $R_m \leqslant 2^{-m} m^{-m}$), so that $R(X)$ has a continuous point derivation of order $\infty$ at 0.

To see that $R(X)$ is regular, let $E \subset X$ be closed, and let $a \in X \sim E$. Then for some $n$, the disc $D_n$ is disjoint from $E$ and contains the point $a$ (consider separately the cases $a \neq 0$, $a = 0$). Thus there is a function $f \in R(X)$ which is nonzero at $a$ and vanishes off $D_n$, hence $f = 0$ on $E$.

## 4. *Related questions*

4.1. This example provides a negative answer to the following question raised by M. Hayashi (private communication):

*Suppose* $f \in R(X)$ *and* $f^{(n)}(a) = 0$ *for* $n = 0, 1, 2, \ldots$.

*Does f vanish identically on the Gleason part of a?*

In this connection, see **[2]**, where Brennan showed that there are some $X$ with no interior for which the answer is yes. In view of Browder's metric density theorem [**3**; p. 176], the part of 0 in our example contains some point $b \neq 0$. There is a function $f$ in $R(X)$ which vanishes on a neighbourhood of 0 and equals 1 at $b$. By any reasonable interpretation of the condition, we have $f^{(n)}(a) = 0$ for all $n$.

A contradiction between this and the results of Wang **[11]** is avoided by the fact that $f$ may have large norm.

4.2. Let $P$ be the Gleason part of 0 in our example. By Melnikov's solution of Wilken's problem **[8]**, $P$ is the only nontrivial part of $R(\text{clos } P)$. Thus there is a regular uniform algebra with just one nontrivial part.

4.3. Our construction can easily be modified to admit a continuous point derivation of infinite order at a finite set of points. Less obvious is the modification to allow continuous point derivations at a set of points having positive area.

Each union of discs $\cup_j D_{nj}$ is contained in an annulus. We can modify the construction, making these annuli thinner, so as to ensure that the union of all the annuli has area less than $\pi$. We can even ensure that the areas of larger annuli, many times thicker than those containing the $D_{nj}$, add up to a number less than $\pi$. This means that the complement in $X$ of the union of these larger annuli has positive area. On this complement, the series

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{r_{nj}}{|z-a_{nj}|^2}$$

will converge, where $a_{nj}$ denotes the centre of $D_{nj}$. This condition is sufficient for the existence of a continuous first-order continuous point derivation at $z$ [**6**].

4.4. A uniform algebra $A$ on $X$ is regular if and only if it is *normal*, in the following sense: given disjoint closed sets $E$, $F$, contained in $X$, there exists $f \in A$ such that $f = 0$ on $E$ and $f = 1$ on $F$ [**1**; p. 58; **10**; p. 345].

### *References*


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