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# FUNCTIONS WITH SMOOTH EXTENSIONS

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ABSTRACT

We remove the nonconstructive hypotheses from the $C^1$ case of Whitney's extension theorem. For a compact set $X \subset \mathbb{R}^d$ we introduce the space $J_c(X)$ of sequentially-calculable bounded point derivations on the algebra $C^1(X)$. We show that if $f$ is continuous on $X$ and $D_u f(a)$ exists and is uniformly continuous for $(a, u) \in J_c(X)$, then $f$ has an extension to $\mathbb{R}^d$ in the class $C^1$.

The object of this paper is to remove the nonconstructive hypotheses from the $C^1$ case of Whitney's extension theorem. We work with real-valued functions on $\mathbf{R}^d$, but the main result goes through for functions mapping any real $C^1$ manifold into a topological vectorspace.

The statement of Whitney's theorem is as follows:

*Let $X$ be a closed subset of $\mathbf{R}^d$, and let $f$ be a continuous function on $X$. Suppose there exist continuous functions $f_1, \ldots f_d$ on $X$ such that, for each point $a \in X$ and each $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$\left| f(y) - f(x) - \sum_{j=1}^{d} (y_j - x_j) f_j(x) \right| \leqq \varepsilon \left| y - x \right|$$

*whenever $x \in X$, $y \in X$, $|x-a| < \delta$, and $|y-a| < \delta$. Then $f$ has a continuously differentiable extension to $\mathbf{R}^d$.*

The only unsatisfactory feature of this result is that the hypotheses involve the functions $f_1, \ldots f_d$, and these are not, in general, uniquely-determined by the values of $f$ on $X$. Thus the result is an extension theorem for 1-jets, not functions. In typical applications, $f$ is given, and the $f_j$'s are not. So there is a need for a result which determines whether or not a given function $f$ has an extension, and which involves only conditions that can be checked by explicit calculation. We get such a result by introducing the bundle of sequentially-calculable bounded point derivations on $X$.

Throughout the paper, $d$ denotes a fixed positive integer, and $X$ denotes a fixed closed subset of $\mathbf{R}^d$. Let $\mathbf{Tan}\ \mathbf{R}^d$ denote the tangent bundle of $\mathbf{R}^d$, that is, the set of ordered pairs $(a, u)$, where $a \in \mathbf{R}^d$ and $u$ is a linear functional mapping $\mathbf{R}^d \to \mathbf{R}$. Alternatively, we may think of $(a, u)$ as a linear functional on the space $C^1$ of continuously differentiable real-valued functions on $\mathbf{R}^d$. For $f \in C^1$ we denote the value of the functional $(a, u)$ at $f$ by $D_u f(a)$. We identify $\mathbf{R}^d$ with its dual, via the usual inner product, so that $\mathbf{Tan}\ \mathbf{R}^d$ is identified with $\mathbf{R}^d \times \mathbf{R}^d$. We denote the projection $(a, u) \to a$ by $\pi$.

The space $C^1$ becomes a Frechet algebra when endowed with the usual topology and pointwise multiplication. The subset

$$I(X) = \{f \in C^1 : f = 0 \text{ on } X\}$$

is a closed ideal in $C^1$, and hence the quotient algebra $C^1(X) = C^1/I(X)$ becomes a Frechet algebra when endowed with the quotient topology. The algebra $C^1(X)$ may be thought of as a space of functions on $X$, namely those functions having $C^1$ extensions. For $a \in X$, let $I_a$ denote the (maximal) ideal in $C^1(X)$ defined by setting $I_a = \{f \in C^1(X) : f(a) = 0\}$. A *bounded point derivation* on the algebra $C^1(X)$ at the point $a \in X$ is an element $L \in C^1(X)^*$ (the space of continuous linear functionals on $C^1(X)$) that annihilates the constants and the ideal $I^2{}_a$. Equivalently, $L$ is an element of $C^1(X)^*$ that satisfies

$$L(fg) = f(a)\, Lg + g(a)\, Lf$$

whenever $f$ and $g$ belong to $C^1(X)$. We denote the space of all bounded point derivations on $C^1(X)$ at the point $a \in X$ by $J(X, a)$, and we define

$$J(X) = \{(a, L) : a \in X,\ L \in J(X, a)\}.$$

Thus $J(X)$ is a subset of $\mathbf{R}^d \times C^1(X)^*$, and is closed in the weak-* topology of the dual of $\mathbf{R}^d \times C^1(X)$. We endow $J(X)$ with the relative topology. The natural surjection $C^1 \to C^1(X)$ induces a continuous injection $J(X) \to J(\mathbf{R}^d) = \mathbf{Tan}\ \mathbf{R}^d$. Furthermore, it is easily seen that the image of $J(X)$ is closed, and since $\mathbf{Tan}\ \mathbf{R}^d$ is locally-compact, it follows that $J(X)$ is homeomorphic to its image, and hence may be regarded as a closed subset of $\mathbf{Tan}\ \mathbf{R}^d$. From this point of view, $J(X, a)$ is a subspace of $\pi^{-1}(a)$. In general, the dimension of $J(X, a)$ may be any integer between 0 and $d$, inclusive.

Let $\mathscr{P}$ denote the linear span of the point masses in $X$, regarded as elements of $C^1(X)^*$. For $L \in \mathscr{P}$, we denote by $\mathrm{spt}L$ the set of points of $X$ that occur in $L$. If $L_n$ is a sequence of elements of $\mathscr{P}$, and $a$ belongs to $X$, then we say that $\mathrm{spt}L_n \to a$ if for each $\varepsilon > 0$ there exists a positive integer $m$ such that $\mathrm{spt}L_n \subset \{y : |y - a| < \varepsilon\}$ whenever $n > m$. We say that an element $L \in C^1(X)^*$ belongs to the space $J_c(X, a)$ if there exists a sequence $L_n \in \mathscr{P}$ converging weak-* to $L$ such that each $L_n$ annihilates the constants, and $\mathrm{spt}L_n \to a$.

**Lemma.** $J_c(X, a) \subset J(X, a)$.

Proof. Suppose $f \in C^1(X)$ and $f \perp J(X, a)$. Then $f \in \mathrm{clos}I_a{}^2$. It is easy to see that $I_a{}^2$ is contained in the closure of the ideal

$$\{g \in C^1(X) : g = 0 \text{ on a neighbourhood of } a\}.$$

Hence $f \perp J_c(X, a)$. The result follows by the separation theorem, since $J(X, a)$ is weak-* closed.

We define $J_c(X)$ as the subset of $J(X)$ given by

$$\{(a, L): a \in X, L \in J_c(X, a)\}.$$

We call $J_c(X)$ the space of *sequentially-calculable bounded point derivations* on $C^1(X)$. We shall see that $J_c(X)$ is always dense in $J(X)$, but in general they need not coincide.

Let $L$ be in $J_c(X, a)$, and let $f$ be a real-valued function on $X$. We say that $Lf$ *exists* and equals $t \in \mathbf{R}$ if

$$\lim_n L_n f$$

exists and equals $t$ whenever $\{L_n\}^{\infty}{}_1 \subset \mathscr{P}$ is a sequence converging weak-* to $L$ such that $L_n 1 = 0$ and $\operatorname{spt} L_n \to a$.

**Theorem.** *Suppose $f$ is a real-valued function on a closed set $X \subset \mathbf{R}^d$, and suppose $D_u f(a)$ exists and is uniformly continuous as a function of $(a, u)$ on $J_c(X)$. Then $f$ has a $C^1$ extension to $\mathbf{R}^d$.*

PROOF. Suppose $f$ satisfies the hypotheses.

We assert that $f$ is continuous on $X$. For otherwise there would exist points $a$ and $a_n$ $(n = 1, 2, 3, \ldots)$ belonging to $X$, such that $a_n \to a$ and $f(a_n) \nrightarrow f(a)$. Passing to a subsequence we could assume that

$$\frac{a_n - a}{|a_n - a|} \to u \in \mathbf{R}^d.$$

Then $u \in J_c(X, a)$, and

$$\frac{g(a_n) - g(a) - |a_n - a| D_u g(a)}{|a_n - a|} \to 0$$

whenever $g \in C^1(X)$; hence by the hypotheses,

$$\frac{f(a_n) - f(a) - |a_n - a| D_u f(a)}{|a_n - a|} \to 0,$$

hence $f(a_n) \to f(a)$, which is a contradiction.

We define $V(a) = \pi^{-1}(a) \cap \operatorname{clos} J_c(X)$, for $a \in X$, and we set $X_j = \{a \in X: \dim V(a) \geqq j\}$, for $j = 0, 1, \ldots, d$. Then each $X_j$ is a closed subset of $X$, and

$$X_d \subset X_{d-1} \subset \ldots \subset X_1 \subset X_0 = X.$$

By adding a remote closed ball to $X$, if necessary, and defining $f = 0$ on this ball, we may ensure that $X_d$ is nonempty. For $a \in X$, let $R_a$ and $S_a$ denote the orthogonal projections of $\pi^{-1}(a) = \mathbf{R}^d$ on $V(a)$ and $V(a)^{\perp}$, respectively.

Define $f^*(a)=f(a)$ for $a \in X$, and $D_u f^*(a)=D_u f(a)$ for $(a, u) \in J_c(X)$. Extend $D_u f^*(a)$ by continuity to $\operatorname{clos} J_c(X)$. Then $D_u f^*(a)$ is defined and continuous on $\pi^{-1}(X_d)$, and satisfies the condition

$$\frac{f^*(x_n)-f^*(y_n)-D_{x_n-y_n} f^*(a)}{|x_n-y_n|} \to 0 \tag{1}$$

whenever $a \in X_d$, $x_n \in X$, $y_n \in X$, $x_n \to a$, and $y_n \to a$.

Construct, in the usual Whitney fashion, a locally-finite covering $\{Q_n\}^{\infty}{}_1$ of $X_{d-1} \sim X_d$ by open cubes $Q_n$ with the side of $Q_n$ comparable to the distance from $Q_n$ to $X_d$. Take corresponding $C^1$ functions $\phi_n$ such that

$$0 \leqq \phi_n \leqq 1,$$
$$\operatorname{spt} \phi_n \subset Q_n,$$
$$\Sigma\, \phi_n = 1 \text{ on } X_{d-1} \sim X_d.$$

For each $n$, take a point $a_n \in X_d$ that is as close as any other point of $X_d$ to $Q_n$. For $(a, u) \in \pi^{-1}(X_{d-1} \sim X_d)$, define

$$D_u f^*(a) = D_{R_a u} f^*(a) + \sum_{n=1}^{\infty} \phi_n(a)\, D_{S_a u} f^*(a_n).$$

For $(a, u) \in V(a)$, we have $R_a u = u$, $S_a u = 0$, so this formula is consistent with the previous definition of $D_u f^*(a)$ in this case. It is easy to check that $D_u f^*(a)$ is continuous on $\pi^{-1}(X_{d-1})$ and satisfies the condition (1) whenever $a \in X_{d-1}$, $x_n \in X$, $y_n \in X$, $x_n \to a$, and $y_n \to a$.

Continuing, we extend $D_u f^*(a)$ in turn to $X_{d-2}, X_{d-3}, \ldots, X_1, X_0 = X$. In the end, $D_u f^*(a)$ is a continuous function on $\pi^{-1}(X)$, and satisfies the condition (1) whenever $a \in X$, $x_n \in X$, $y_n \in X$, $x_n \to a$, and $y_n \to a$. The classical Whitney extension formula (Stein, p. 177, (18)) then yields an extension of $f^*$ to $R^d$ that belongs to the class $C^1$.

**Corollary.** *$J_c(X)$ is dense in $J(X)$, that is, each bounded point derivation on $C^1(X)$ is the weak-* limit of a sequence of sequentially-calculable bounded point derivations.*

PROOF. Suppose $f$ belongs to $C^1(X)$, and $D_u f(a)=0$ whenever $(a, u) \in J_c(X)$. Then the extension $f^*$ constructed in the proof of the theorem satisfies $D_u f^*(a)=0$ whenever $(a, u) \in \pi^{-1}(X)$. Thus $D_u f(a) = D_u f^*(a) = 0$ whenever $(a, u) \in J(X)$. Since the topology on $J(X)$ is the weak-* topology, the result follows by the separation theorem.

## Reference

STEIN, E. M. 1970 *Singular Integrals and Differentiability Properties of Functions.* Princeton.