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METAHARMONIC APPROXIMATION IN LIPSCHITZ NORMS*

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Abstract

We study approximation in non-integral Lipschitz norms by harmonic, biharmonic, and other metaharmonic functions. We prove that the approximation problems involved are local, and we relate approximation problems in different norms to one another.

Let d be an integer greater than 2. For a compact set $X \subset \mathbb{R}^d$ and a positive integer p, $H_p(X)$ denotes the space of real-valued C^∞ functions f on $\mathbb{R}^d$ which satisfy the equation

$$\triangle^p f = 0$$

on a neighbourhood of X; here $\triangle$ is the Laplacian. This paper concerns the closure of $H_p(X)$ in Lip(β, X), for positive nonintegral β (cf. §1 for the definition of Lip(β, X)). In earlier papers [1, 2] the author set up a scheme for attacking approximation problems concerning rational functions in the plane. Some parts of this scheme carry over to any context in which the appropriate integral operator has certain continuity properties. For $H_p(X)$ the appropriate operator is the Newtonian potential, and the required bounds are obtained in Lemmas 1 and 2.

Here is a summary of the results. In all cases, X is a compact subset of $\mathbb{R}^d$, $d \geqq 3$, and p is a positive integer.

Theorem 1. *Let* $0 < \beta \notin \mathbb{Z}$. *Then the closure of* $H_p(X)$ *in* Lip(β, X) *is locally determined, in the sense that if f is a function on X, and each point of X has a closed neighbourhood U such that f belongs to the closure of* $H_p(X \cap U)$ *in* Lip$(\beta, X \cap U)$, *then f belongs to the closure of* $H_p(X)$ *in* Lip(β, X).

This partially complements a result of Weinstock [4, Proposition 5]. He proved that the closure of $H_A(X)$ in the germ space $C^k(X)$ is local, whenever A is an elliptic

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operator with constant coefficients, and k is a nonnegative *integer* less than the degree of A. Here $H_A(X)$ denotes the space of C^∞ functions which satisfy the equation $Af=0$ on a neighbourhood of X.

Theorem 2. *Suppose that every real-valued continuous function on X is the uniform limit of a sequence of functions in $H_p(X)$. Then the closure of $H_{p+1}(X)$ in* $\mathrm{Lip}(\beta, X)$ *is* $\mathrm{lip}(\beta, X)$, *for* $0<\beta<2$.

Theorem 3. $0<\beta\notin\mathbb{Z}$. *Consider the two statements:*

(1) *The closure of $H_p(X)$ in* $\mathrm{Lip}(\beta, X)$ *is* $\mathrm{lip}(\beta, X)$.

(2) *The closure of $H_{p+1}(X)$ in* $\mathrm{Lip}(\beta+2, X)$ *is* $\mathrm{lip}(\beta+2, X)$.

Let m be the largest integer less than β. If X is uniformly m-thick, then (1) *implies* (2). *If X is uniformly $(m+2)$-thick, then* (2) *implies* (1).

The concept of m-thickness is defined in §3 (cf. also [1]).

Theorem 4. *Suppose X has zero d-dimensional Lebesgue measure. Then $H_1(X)$ is dense in* $\mathrm{lip}(\beta, X)$ *in* $\mathrm{Lip}(\beta, X)$ *norm, for* $0<\beta<2$.

In case $0<\beta\leqq 1$, this is contained in Proposition 3 of [4].

§1. For $3\leqq d\in\mathbb{Z}$, $\mathscr{E}$ denotes the space of real-valued infinitely differentiable functions on $\mathbb{R}^d$, and $\mathscr{D}$ denotes the space of functions in $\mathscr{E}$ with compact support. For a multi-index $i=(i_1, \ldots, i_m)\in\mathbb{Z}^d_+$, the corresponding partial derivative of a function f at a point $a\in\mathbb{R}^d$ is denoted $D_if(a)$. The *order* of this derivative is $|i|=\Sigma i_m$. For $\beta>0$ we say that a bounded continuous real-valued function f, defined on $\mathbb{R}^d$, belongs to Lip β if

(1) D_if is continuous and bounded for $|i|<\beta$,

(2) D_if satisfies a Lipschitz condition with index $\alpha=\beta-|i|$ for $\beta-1\leqq|i|<\beta$:

$$\sup_{x,y}\frac{|D_if(x)-D_if(y)|}{|x-y|^\alpha}\leqq M_i<\infty.$$

When endowed with the norm

$$\|f\|_\beta=\sum_{|i|<\beta}\|D_if\|_u+\sum_{\beta-1\leq|i|<\beta}\text{least } M_i,$$

Lip β becomes a Banach algebra (here $\|\cdot\|_u$ stands for the uniform norm on $\mathbb{R}^d$). The closed subalgebra lip β consists of those $f\in$ Lip β such that

$$\sup_{|x-y|\leq\delta}\frac{|D_if(x)-D_if(y)|}{|x-y|^\alpha}\downarrow 0$$

as $\delta\downarrow 0$, for $\beta-1\leqq|i|<\beta$. If $X\subset\mathbb{R}^d$ is compact, then the set

$$I(X)=\{f\in \mathrm{Lip}\,\beta : f=0 \text{ on } X\}$$

is a closed ideal, and hence the quotient

$$\mathrm{Lip}(\beta, X)=\mathrm{Lip}\,\beta/I(X)$$

becomes a Banach algebra when given the quotient norm. Clearly $\mathrm{Lip}(\beta, X)$ may be thought of as a space of functions on X (in these terms $\mathrm{Lip}(\beta, X)$ consists of those functions on X which have extensions in $\mathrm{Lip}\,\beta$, and the norm of a function f in $\mathrm{Lip}(\beta, X)$ is $\|f\|_{\beta,X}=\inf\{\|g\|_\beta : g\in \mathrm{Lip}\,\beta, f=g \text{ on } X\}$).

The set

$$\mathrm{lip}(\beta, X)=\frac{\mathrm{lip}\,\beta+I(X)}{I(X)}$$

forms a closed subalgebra of $\mathrm{Lip}(\beta, X)$. For $0<\beta\notin\mathbb{Z}$, $\mathscr{D}(+I(X))$ is dense in $\mathrm{lip}(\beta, X)$. If $0<\beta<\gamma$, $\beta\notin\mathbb{Z}$, then each function in $\mathrm{Lip}(\gamma, X)$ belongs to $\mathrm{lip}(\beta, X)$. Membership in $\mathrm{Lip}(\beta, X)$ is a local property [2], and the same holds for $\mathrm{lip}(\beta, X)$.

$\mathscr{L}^d$ denotes Lebesgue measure on $\mathbb{R}^d$. The Newtonian potential of a function $f\in\mathscr{D}$ is the function $Pf\in\mathscr{E}$ given by

$$Pf(x)=-\lambda\int\frac{f(y)}{|y-x|^{d-2}}\,d\mathscr{L}^d y \quad (x\in\mathbb{R}^d),$$

where

$$\lambda=\frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{d/2}}.$$

As is well known [3, pp. 117–118], the operator P inverts the Laplacian, $\triangle$, in the sense that

$$\triangle Pf=P\triangle f=f\,(f\in\mathscr{D}).$$

Theorems 1-3 hinge upon the continuity properties of P with respect to the various Lipschitz norms and the uniform norm. These properties are set out in the first two lemmas.

Lemma 1. *Suppose* $0<\beta\notin\mathbb{Z}$. *There is a constant* K_1, *which depends only on* β, d, *and* diam spt f, *such that*

$$\|Pf\|_{\beta+2}\leq K_1\|f\|_\beta,$$

whenever $f\in\mathscr{D}$.

PROOF. For $0<\beta<1$, this fact is implicit in [3]. In fact, the space Λ_β of Stein coincides with $\mathrm{Lip}\beta$ for $0<\beta<1$ [3, p. 150]. Hence there is a constant $K_2>0$, depending only on d and β, such that for $f\in\mathscr{D}$,

$$\|R_j f\|_\beta\leq K_2\|f\|_\beta\,(1\leq j\leq d),$$

where R_j denotes the j-th Riesz transform [3, p. 143]. Thus, if $1\leq i, j\leq d$, then

$$\begin{aligned}\|D_{ij}Pf\|_\beta &=\|R_i R_j\triangle Pf\|\beta\\ &=\|R_i R_j f\|_\beta\\ &\leq K^2{}_2\|f\|_\beta\,(f\in\mathscr{D}).\end{aligned}$$

Since $D_{ij}Pf(x)$ decays like $\mathrm{dist}(x, \mathrm{spt}\,f)^{-d}$ as $|x|\uparrow\infty$, we deduce successively that the lower-order norms

$$\| D_{ij}\, Pf \|_u, \ \| D_i\, Pf \|_u, \ \| Pf \|_u$$

are similarly bounded.

We obtain the result for higher nonintegral β by induction. The key fact is that P commutes with partial derivatives:

$$\begin{aligned}[D_i Pf](x) &= \frac{\partial}{\partial x_i}\left\{-\lambda \int \frac{f(y)}{|y-x|^{d-2}}\, d\mathscr{L}^d y\right\} \\ &= \frac{\partial}{\partial x_i}\left\{-\lambda \int \frac{f(u+x)}{|u|^{d-2}}\, d\mathscr{L}^d u\right\} \\ &= -\lambda \int \frac{D_i f(u-x)}{|u|^{d-2}}\, d\mathscr{L}^d u \\ &= -\lambda \int \frac{D_i f(y)}{|y-x|^{d-2}}\, d\mathscr{L}^d y \\ &= [Pd_i f](x)\ (x \in \mathbb{R}^d, f \in \mathscr{D}).\end{aligned}$$

Suppose $0<\alpha<1$, $0 \leqq m \in \mathbb{Z}$, and the result is true for $\beta = m+\alpha$. We will prove it for $\beta = m+1+\alpha$.

Let $f \in \mathscr{D}$, let $j \in \mathbb{Z}_+^{m+2}$ be a multi-index, and let $i=(j,k) \in \mathbb{Z}_+^{m+3}$ be another. We have

$$\begin{aligned}\| D_i Pf \|_\alpha &= \| D_j\, PD_k f \|_\alpha \\ &\leqq K_3\, \| D_k f \|_{m+\alpha}, \text{ by hypothesis} \\ &\leqq K_4\, \| f \|_{m+1+\alpha}.\end{aligned}$$

It follows that the result is true for $\beta = m+1+\alpha$.

Lemma 2. *Suppose* $0<\beta<2$. *There is a constant* K_5, *which depends only on* d, β *and* diam spt f, *such that*

$$\| Pf \|_\beta \leqq K_5\, \| f \|_u\ (f \in \mathscr{D}).$$

Proof. Since Pf decays as it does, it suffices to prove the case $1<\beta<2$.

Let $\alpha = \beta - 1$. The Riesz potentials I_τ are defined by setting [3, p. 117]

$$I_\tau f(x) = \frac{1}{\gamma(\tau)} \int \frac{f(y)}{|y-x|^{d-\tau}}\, d\mathscr{L}^d y$$

whenever $x \in \mathbb{R}^d$ and $f \in \mathscr{D}$, where

$$\gamma(\tau) = \frac{\pi^{d/2}\, 2^\tau\, \Gamma\left(\frac{\tau}{2}\right)}{\Gamma\left(\frac{n-\tau}{2}\right)}.$$

Observe that $P = -I_2$. By comparing the Fourier multipliers corresponding to the various terms, we obtain the equality

$$D_j Pf = R_j I_1 f\ (f \in \mathscr{D}, 1 \leqq j \leqq d).$$

Thus for $f \in \mathscr{D}$,

$$\begin{aligned}\| D_j Pf \|_\alpha &= \| R_j I_1 f \|_\alpha \\ &\leqq K_2 \|I_1 f\|_\alpha \\ &\leqq K_6 \|f\|_u \qquad (*).\end{aligned}$$

The last inequality is justified by making a direct estimate of the Lip α norm of the function

$$g(x) = \int \frac{f(y)}{|y-x|^{d-1}}\, d\mathcal{L}^d y,$$

as follows.

Let diam spt $f = R$, and let $x, z \in \mathbb{R}^d$. Set $r = \frac{1}{2}|x-z|$. Clearly,

$$\| g \|_u \leqq K_7 \| f \|_u R,$$

where K_7 is the surface area of the unit sphere in $\mathbb{R}^d$. Hence, if $r \geqq R$, we have

$$\frac{|g(x)-g(z)|}{|x-z|^\alpha} \leqq 2^{1-\alpha} R^{1-\alpha} K_7 \| f \|_u .$$

So suppose $r < R$, and let $B_1 = \{w \in \mathbb{R}^d : |x-w| \leqq r\}$,
$B_2 = \{w \in \mathbb{R}^d : |z-w| \leqq r\}$.

Then

$$\begin{aligned}|g(x)-g(z)| &\leqq \int \left| \frac{1}{|y-x|^{d-1}} - \frac{1}{|y-z|^{d-1}} \right| d\mathcal{L}^d y \, \| f \|_u \\ &\leqq K_8 \sum_{j=1}^{d-1} r \int \frac{d\mathcal{L}^d y}{|y-x|^{d-j} |y-z|^j} \\ & K_8 \sum_{j=1}^{d-1} \left\{ r \int_{B_1} + r \int_{B_2} + r \int_{\mathbb{R}^d \setminus B_1 \setminus B_2} \right\}.\end{aligned}$$

Now

$$\begin{aligned} r \int_{B_1} \frac{d\mathcal{L}^d y}{|y-x|^{d-j} |y-z|^j} &\leqq K_9 r^{1-j} \int_0^r s^{j-1}\, ds \\ &= K_{10} r \\ &\leqq K_{11} r^\alpha .\end{aligned}$$

Also,

$$r \int_{B_2} \leqq K_{11} r^\alpha .$$

Next,

$$\begin{aligned} & r \int_{\mathbb{R}^d \setminus B_1 \setminus B_2} \frac{d\mathcal{L}^d y}{|y-x|^{d-j} |y-z|^j} \\ & \leqq r \int_{\mathbb{R}^d \setminus B_1} \frac{d\mathcal{L}^d y}{|y-x|^d} + r \int_{\mathbb{R}^d \setminus B_2} \frac{d\mathcal{L}^d y}{|y-z|^d}\end{aligned}$$

$$\leqq K_{12}\, r \int_r^R \frac{ds}{s}$$
$$\leqq K_1{}^2\, r^\alpha \int_0^R \frac{ds}{s^\alpha}$$
$$\leqq K_{13}\, r^\alpha .$$

Thus

$$|\, g(x)-g(z)\, | \leqq K_{14} r^\alpha .$$

Hence, no matter what points x and z are taken,

$$|\, g(x)-g(z)\, | \leqq K_{15}\, |\, x-z\, |^\alpha ,$$

where K_{15} depends only on d, α and diam spt f.

The inequality (*) implies that

$$\|\, Pf\, \|_\beta \leqq K_5\, \|\, f\, \|_\alpha ,$$

as required.

§2. This section is devoted to the proof of Theorem 1. The main tool in the proof is the localisation operator $L_{\phi,p}$. Fix $X \subset \mathbb{R}^d$, compact, and choose $\psi \in \mathscr{D}$ such that $\psi \equiv 1$ on a neighbourhood of X. Define

$$Cf = P(\psi f)\, ,$$
$$L_{\phi,p} f = C^{p-1} P(\phi .\, \triangle^p f)\, ,$$

whenever $f \in \mathscr{D}$, $\phi \in \mathscr{E}$, $1 \leqq p \in \mathbb{Z}$. It is a simple matter to check that

$$\triangle^p L_{\phi,p} f = \phi .\, \triangle^p f$$

on the interior of the set $\psi^{-1}(1)$. Thus, inside this neighbourhood of X, $L_{\phi,p} f$ satisfies the equation

$$\triangle^p g = 0$$

wherever f does, and off the support of ϕ. Also

$$\triangle^p (L_{\phi,p} f - f) = 0$$

on the interior of $\psi^{-1}(1) \cap \phi^{-1}(1)$. Clearly, $L_{\phi,p}$ is the analogue in this situation of the famous Vitushkin localisation operator T_ϕ for analytic functions.

Lemma 3. *Let $1 \leqq p \in \mathbb{Z}$, $0 < \beta \notin \mathbb{Z}$, and $\phi \in \mathscr{D}$. Then there is a constant K_{16}, depending on d, β, p, ϕ and ψ, but not on f, such that*

$$\|\, L_{\phi,p} f\, \|_\beta \leqq K_{16}\, \|\, f\, \|_\beta$$

whenever $f \in \mathscr{D}$.

Proof. Consider first the case $p=1$, $\beta > 2$. Applying Lemma 1 we obtain

$$\|\, L_{\phi,p} f\, \|_\beta = \| P(\phi .\, \triangle f) \|_\beta$$
$$\leqq K_1\, \|\, \phi .\, \triangle f\, \|_{\beta-2}$$
$$\leqq K_1\, \|\, \phi\, \|_{\beta-2}\, \|\, \triangle f\, \|_{\beta-2}$$
$$\leqq K_{17}\, \|\, f\, \|_\beta .$$

Next, consider the case $p=1$, $0<\beta<2$. Here, and in the remainder of the proof, we adopt the convention that each operator in a string shall be understood to act on the whole string to its right. For instance,

$$R_jP(\nabla\phi).\nabla f.g \text{ means } R_j(P((\nabla\phi).(\nabla(f.g)))).$$

we have

$$\| L_{\phi,p}f \|_\beta = \| P\{\triangle\phi.f - 2\nabla\phi.\nabla f - f.\triangle\phi\} \|_\beta$$
$$\leqq \| \phi.f \|_\beta + 2\| P(\nabla\phi).\nabla f \|_\beta + \| f.\triangle\phi \|_\beta.$$

Only the second term poses a problem. To estimate it we set $\nabla\phi = (\phi_1, \ldots, \phi_d)$, and proceed as follows.

(*a*) If $0<\beta<1$, we infer from the identity

$$P\phi_j.D_jf = PD_j\phi_j.f - Pf.D_j\phi_j$$
$$= -R_jI_1\phi_j.f - Pf.D_j\phi_j$$

that

$$\| P\phi_j.D_jf \|_\beta \leqq \| R_jI_1\phi_j.f \|_\beta + \| Pf.D_j\phi_j \|_\beta$$
$$\leqq K_6 \| \phi_j.f \|_u + K_5 \| f.D_j\phi_j \|_u$$
$$\leqq K_{18} \| f \|_u$$
$$\leqq K_{18} \| f \|_\beta .$$

Here we have made use of Lemma 2.

(*b*) If $1<\beta<2$, another application of Lemma 2 yields

$$\| P\phi_j.D_jf \|_\beta \leqq K_5 \| \phi_j.D_jf \|_u$$
$$\leqq K_5 \| \phi_j \|_u \| f \|_\beta .$$

Thus the lemma is proved for $p=1$ and all β. To get the result for $p>1$ we proceed by induction. The induction hypothesis is as follows:

($\mathscr{H}_p$): *If* $0<\beta\notin\mathbb{Z}$ *and* $\psi_1, \psi_2, \ldots, \psi_p \in \mathscr{D}$, *then there exists a constant* K_{19}, *which does not depend on* f, *such that*

$$\| P\psi_1.P\psi_2.\ldots P\psi_p.\triangle^p f \|_\beta \leqq K_{19} \| f \|_\beta.$$

Certainly ($\mathscr{H}_1$) holds. Now suppose ($\mathscr{H}_k$) holds for $1\leqq k\leqq p$. We will prove that ($\mathscr{H}_{p+1}$) holds.

Let $0<\beta\notin\mathbb{Z}$, and let $\psi_1, \ldots, \psi_{p+1}\in\mathscr{D}$. For $f\in\mathscr{D}$ we have

$$\psi_{p+1}.\triangle^{p+1}f = \triangle\psi_{p+1}.\triangle^p f - (\triangle\psi_{p+1}).\triangle^p f - (\nabla\psi_{p+1}).\nabla\triangle^p f,$$

hence

$$P\psi_1.P\psi_2.\ldots P\psi_{p+1}\triangle^{p+1}f$$
$$= P\psi_1\ldots P\psi_p.\psi_{p+1}.\triangle^p f - P\psi_1\ldots P\psi_p.P(\triangle\psi_{p+1}).\triangle^p f$$
$$- P\psi_1\ldots P\psi_p P(\nabla\psi_{p+1}).\triangle^p\nabla f$$
$$= Z_1 - Z_2 - Z_3.$$

By ($\mathscr{H}_p$) there exist constants K_{20} and K_{21}, depending on $\psi_1, \ldots, \psi_{p+1}$, but not on f, such that

$$\| Z_1 \|_\beta \leqq K_{19} \| f \|_\beta ,$$
$$\| Z_2 \|_\beta \leqq K_{20} \| f \|_\beta .$$

To estimate Z_3 we consider three cases.

(*a*) $\beta > 2$. By Lemma 1 and $(\mathscr{H}_p)$ we have

$$\| Z_3 \|_\beta \leqq \sum_{j=1}^{d} \| P\psi_1 \cdot \ldots P\psi_p . P(D_j\psi_{p+1}) . \triangle^p D_j f \|_\beta$$

$$\leqq \sum_{j=1}^{d} K_{21} \| P\psi_2 \cdot \ldots P\psi_p . P(D_j\psi_{p+1}) . \triangle^p D_j f \|_{\beta-2}$$

$$\leqq \sum_{j=1}^{d} K_{22} \| D_j f \|_{\beta-2}$$

$$\leqq K_{23} \| f \|_{\beta-1} .$$

(*b*) $1 < \beta < 2$. Using Lemma 2 in place of Lemma 1 we obtain

$$\| Z_3 \|_\beta \leqq K_{24} \| f \|_1 \leqq K_{24} \| f \|_\beta .$$

(*c*) $0 < \beta < 1$.

$$P\psi_1 \cdot \ldots P\psi_p . P(D_i\psi_{p+1}) . \triangle^p D_i f$$
$$= P\psi_1 \cdot \ldots P\psi_p . PD_i(D_i\psi_{p+1}) . \triangle^p f$$
$$- P\psi_1 \cdot \ldots P\psi^p . P(D_i^2\psi_{p+1}) . \triangle^p f .$$

The second term is suitably bounded, by $(\mathscr{H}_p)$, and the first equals

$$P\psi_1 \cdot \ldots P\psi_p . D_i P(D_i\psi_{p+1}) . \triangle^p f$$
$$= P\psi_1 \cdot \ldots PD_i\psi_p . P(D_i\psi_{p+1}) . \triangle^p f$$
$$- P\psi_1 \cdot \ldots P(D_i\psi_p) . P(D_i\psi_{p+1}) . \triangle^p f .$$

The second term is again suitably bounded, and the first can again be broken up. Continuing, we find that Z_3 differs from a suitably bounded quantity by

$$D_i P\psi_1 \cdot \ldots P\psi_p . P(D_i\psi_{p+1}) . \triangle^p f .$$

But

$$\| D_i P\psi_1 \cdot \ldots P\psi_p . P(D_i\psi_{p+1}) . \triangle^p f \|_\beta$$
$$\leqq \| P\psi_1 \cdot \ldots P\psi_p . P(D_i\psi_{p+1}) . \triangle^p f \|_{\beta+1}$$
$$\leqq K_5 \| P\psi_2 \cdot \ldots P\psi_p . P(D_i\psi_{p+1}) . \triangle^p f \|_u$$

by Lemma 2,

$$\leqq K_5 \| P\psi_2 \cdot \ldots P\psi_p . P(D_i\psi_{p+1}) . \triangle^p f \|_\beta$$
$$\leqq K_{25} \| f \|_\beta ,$$

hence

$$\| Z_3 \|_\beta \leqq K_{26} \| f \|_\beta .$$

The induction step is complete, and $(\mathscr{H}_p)$ holds for every positive integer p. To see that $(\mathscr{H}_p)$ implies the statement of the Lemma, take

$$\psi_1 = \psi_2 = \ldots = \psi_{p-1} = \psi,$$
$$\psi_p = \phi .$$

PROOF OF THEOREM 1. Let f be a function on X, and suppose $\{U_1, \ldots, U_n\}$ is a finite covering of X by closed balls such that $f | X \cap U_j$ is approximable in Lip(β, $X \cap U_j$) norm by functions in $H_p(X \cap U_j)$.

Since Lip(β, X) is locally determined, f has an extension in Lip β. Let us denote such an extension by the same symbol f. We may assume spt f is compact. Choose functions $\phi_1, \phi_2, \ldots, \phi_n \in \mathscr{D}$ so that

(1) spt $\phi_j \subset U_j$,

(2) $\Sigma\phi_j = 1$ on a neighbourhood of X.

Then the function

$$f_1 = f - \sum_{j=1}^{n} L_{\phi_j,p} f$$
$$= f - L_{\Sigma\phi_j,p} f$$

satisfies $\triangle^p f_1 = 0$ on a neighbourhood of X. By Lemma 3, $L_{\phi_j,p} f \in$ Lip β, and thus f_1 belongs to the intersection of $H_p(X)$ and Lip(β, X).

Fix j, $1 \leqq j \leqq n$, and choose a sequence $\{f_m\}_1^\infty \subset H_p(X \cap U_j)$ such that $\| f - f_m \|_{\beta, X \cap U_j}$ tends to zero. By modifying f_m off a neighbourhood of $X \cap U_j$ we may assume that $\| f - f_m \|_\beta$ tends to zero.

By Lemma 3, we infer that

$$\| L_{\phi_j,p} f_m - L_{\phi_j,p} f \|_\beta \to 0.$$

Since

$$\triangle^p L_{\phi_j,p} f_m(x) = 0$$

holds whenever $x \in$ int $\psi^{-1}(1)$ and $\phi_j(x).\triangle^p f_m(x) = 0$, we see that $L_{\phi_j,p} f_m \in H_p(X)$, hence

$$f = \lim_{m\to\infty} \{f_1 + \Sigma L_{\phi_j,p} f_m^{(j)}\}$$

belongs to the closure of $H_p(X)$ in Lip(β, X).

§3. In this section we prove Theorems 2 and 3.

The dual space of Lip(β, X), denoted Lip(β, X)*, consists of all continuous linear functionals on Lip(β, X). Since the natural mapping $f \mapsto f + I(X)$ of $\mathscr{E} \to$ Lip(β, X) is continuous, any element $T \in$ Lip(β, X)* induces an element of $\mathscr{E}'$ – a distribution with compact support. We normally use the same symbol T for this distribution, but when we wish to distinguish we use $T | \mathscr{E}$. Observe that the equality $T | \mathscr{E} = S | \mathscr{E}$ does not imply that $T = S$, indeed $T | \mathscr{E} = 0$ if and only if $T \perp$ lip(β, X). The support of $T | \mathscr{E}$ is a subset of X, since $T | \mathscr{E}$ annihilates $\mathscr{D} \cap I(X)$. The Newtonian potential P provides a continuous injection of $\mathscr{D}$ into $\mathscr{E}$, hence if $T \in \mathscr{E}'$, then $TP \in \mathscr{D}'$. If $T \in \mathscr{E}' \cap H_1(X)^\perp$, then spt $T \subset X$, since every C^∞ function which vanishes on a neighbourhood of X lies in $H_1(X)$.

Lemma 4. *Let $T \in \mathscr{E}'$ and let $X \subset \mathbb{R}^d$ be compact. Then $T \perp H_1(X)$ if and only if* spt $TP \subset X$.

PROOF. Suppose $T \perp H_1(X)$. Let $\phi \in \mathscr{D}$, $X \cap \operatorname{spt} \phi = \varnothing$. Then $P\phi \in H_1(X)$, hence $TP\,\phi = 0$. Thus spt $TP \subset X$.

Conversely, suppose spt $TP \subset X$, and let $h \in H_1(X)$. Choose $\phi \in \mathscr{D}$ such that $\phi = 1$ on a neighbourhood of X. Then

$$Th = T\phi.h = TP \triangle \phi.h = 0,$$

since $\triangle\phi h = \triangle h = 0$ on a neighbourhood of spt TP.

Lemma 5. *Let $T \in \mathscr{E}'$. Then $TP \in \mathscr{E}' \cap H_p(X)^\perp$ if and only if $T \perp H_{p+1}(X)$.*

PROOF. Suppose $TP \in \mathscr{E}' \cap H_p(X)^\perp$. Fix $f \in H_{p+1}(X) \cap \mathscr{D}$. Then $\triangle f \in H_p(X)$, hence

$$Tf = TP \triangle f = 0.$$

Thus $T \perp H_{p+1}(X)$.

Conversely, suppose $T \perp H_{p+1}(X)$. Fix $f \in H_p(X) \cap \mathscr{D}$.

Then

$$\triangle^{p+1} Pf = \triangle^p f = 0$$

on a neighbourhood of X, hence $TPf = 0$. Thus $TP \perp H_p(X) \cap \mathscr{D}$. By Lemma 4, $TP \in \mathscr{E}'$, hence by continuity $TP \perp H_p(X)$.

PROOF OF THEOREM 2. Suppose $X \subset \mathbb{R}^d$ is compact, $0 < \beta < 2$, $1 \leqq p \in \mathbb{Z}$, and $H_p(X)$ is uniformly dense in $C(X)$ (the space of all real-valued continuous function on X).

We wish to show that $H_{p+1}(X)$ is dense in lip(β, X) in Lip(β, X) norm. We will use the separation theorem.

Let $T \in \text{Lip}(\beta, X)^* \cap H_{p+1}(X)^\perp$. We will show that $T \mid \mathscr{E} = 0$, so that $T \perp \text{lip}(\beta, X)$.

By Lemma 5, $TP \perp H_p(X)$, and by Lemma 4, the support of TP is contained in X. We claim that TP is a distribution of order zero:

$$|TPf| \leqq K_{27} \|f\|_{u,X} \; (f \in \mathscr{E}),$$

where K_{27} does not depend on f. To see this, fix $f \in \mathscr{E}$ and $1 > \varepsilon > 0$. Choose $g \in \mathscr{D}$ with support contained in the ε-neighbourhood of X, with $f = g$ on a neighbourhood of X, and

$$\|g\|_u \leqq \|f\|_{u,X} + \varepsilon.$$

Then by Lemma 2,

$$\|Pg\|_\alpha \leqq K_5 \|g\|_u,$$

where K_5 depends only on diam X. Thus

$$\begin{aligned} |TPf| &= |TPg| \\ &\leqq \|T\|_\alpha K_5 \|g\|_u \\ &\leqq K_{28} (\|f\|_{u,X} + \varepsilon). \end{aligned}$$

The claim follows. Hence $TP \in C(X)^* \cap H_p(X)^\perp$, and hence by the separation theorem and the hypothesis, $TP \mid \mathscr{E} = 0$. But $P\mathscr{D} \supset P\triangle\mathscr{D} = \mathscr{D}$, hence $T \perp \mathscr{D}$, hence $T \perp \mathscr{E}$.

Before discussing Theorem 3, we must define m-thickness [1].

All sets are considered 0-thick.

Let $1 \leqq m \in \mathbb{Z}$, and let $X \subset \mathbb{R}^d$ be compact. Let $D^m(X)$ be the closure of $\mathscr{E}+I(X)$ in $\mathrm{Lip}(m, X)$. For $a \in X$, $J_m(X, a)$ denotes the space of m-th *order differential operators* on the Banach algebra $D_m(X)$ at a, i.e. $J_m(X, a)$ consists of all those functionals $T \in D^m(X)^*$ which annihilate the subspace generated by the constants and the $(m+1)$-st. power of the maximal ideal at a.

At isolated points of X, $J_m(X, a)=\{0\}$. At accumulation points, the dimension of $J_m(X, a)$ lies between m and $\frac{1}{2}m(m+3)$, and either extreme value may occur. We say X is *m-thick* if

$$\dim J_m(X, a)=\tfrac{1}{2}m(m+3)$$

for each $a \in X$. If this is the case, then all the partial derivatives

$$f \mapsto D_i f(a) \quad (f \in \mathscr{E})$$

corresponding to multi-indices i with $|i| \leqq m$ and points $a \in X$, extend to unique elements of $J_m(X, a)$. We denote the extensions by the same symbols. We say X is *uniformly m-thick* if there exists a constant $K_{29}>0$ such that

$$|D_i f(a)| \leqq K_{29} \|f\|_{m,X} \qquad (*)$$

whenever $|i| \leqq m, f \in \mathscr{E}$ and $a \in X$. Uniform m-thickness implies m-thickness, and hence implies that (*) holds for $f \in D^m(X)$. By an approximation argument one sees that if X is uniformly m-thick, then $D_i f(a)$ is a continuous function of a on X, for each fixed $f \in D^m(X)$ and $|i| \leqq m$.

Whenever we wish to distinguish, we will write $\tilde{f}$ for the coset $f+I(X)$ corresponding to a function f in Lip β.

PROOF OF THEOREM 3. Suppose $X \subset \mathbb{R}^d$ is compact, $1 \leqq p \in \mathbb{Z}$, $0<\beta \notin \mathbb{Z}$. Let m be the largest integer less than β, and suppose X is uniformly m-thick. Suppose $H_p(X)$ is dense in $\mathrm{lip}(\beta, X)$. We will prove that $H_{p+1}(X)$ is dense in $\mathrm{lip}(\beta+2, X)$.

Let $T \in \mathrm{Lip}(\beta+2, X)^* \cap H_{p+1}(X)^{\perp}$. We wish to show that $T|\mathscr{E}=0$.

The support of TP lies in X, and $TP \perp H_p(X)$, by Lemmas 4 and 5. By Lemma 1,

$$|TPf| \leqq \|T\|_{\beta+2} K_1 \|f\|_\beta \, (f \in \mathscr{D}),$$

where K_1 depends only on d, β and diam X. Thus TP has a unique extension (also denoted TP) in $(\mathrm{lip}\,\beta)^*$.

Suppose f and g are two functions in lip β such that $\tilde{f}=\tilde{g}$, i.e. $f=g$ on X. Let $\varepsilon>0$ be given. Since X is uniformly m-thick, we have

$$D_i f(a)=D_i g(a)$$

whenever $|i| \leqq m$ and $a \in X$. Let $h=f-g$. For each multi-index $i \in \mathbb{Z}_+^d$ with $|i| \leqq m$, set $h^{(i)}=D_i h$. Define $R_i(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ by setting [3, p. 176]

$$h^{(i)}(x)=\sum_{|i+j| \leqq m} \frac{h^{(i+j)}(y)}{j!}(x-y)^j+R_i(x, y)$$

whenever $x, y \in \mathbb{R}^d$, where

$$j!=\prod_{k=1}^{d} j_k!,$$

$$(x-y)^j = \prod_{k=1}^{d} (x_k - y_k)^{jk}.$$

Since $h^{(i)}=0$ on X, $R_i=0$ on $X \times X$ for $|i| \leqq m$, and $h \in \text{lip}\ \beta$, it follows that there is a compact neighbourhood Y of X such that

$$|h^{(i)}(x)| \leqq \varepsilon\ (x \in Y, |i| \leqq m),$$
$$|R_i(x,y)| \leqq \varepsilon |x-y|^{\beta-|i|}\ (x, y \in Y, |i| \leqq m).$$

Hence, by the Whitney-Calderón-Zygmund Extension Theorem [3, p. 177, Theorem 4 and p. 194, § 4.6], there exists a function $k \in \text{lip}\ \beta$ such that $h=k$ on Y and $\|k\|_\beta < 2^{m+1}\varepsilon$. Thus

$$|TP(f-g)| = |TPk| \leqq K_{30}\varepsilon.$$

Since $\epsilon > 0$ was arbitrary, $TPf = TPg$. Hence TP is a continuous linear functional on lip β with

$$I(X) \cap \text{lip}\ \beta \subset \ker TP.$$

By the Hahn-Banach theorem, there exists an extension Q of TP in $(\text{Lip}\ \beta)^*$ with $I(X) \subset \ker Q$. It follows that Q induces a continuous linear functional R on $\text{Lip}(\beta, X)^*$.

If $f \in H_p(X)$, then $R\tilde{f} = Qf = TPf = 0$, hence $R \perp H_p(X) + I(X)$. Since $H_p(X) + I(X)$ is dense in $\text{lip}(\beta, X)$, it follows that $R \perp \text{lip}(\beta, X)$, hence $Q \perp \text{lip}\ \beta$, hence $TP \perp \mathscr{E}$, hence $T \perp \mathscr{E}$.

The other direction of the theorem is similar.

§4. In this section we prove Theorem 4. The reader will observe that there is a representation formula for $TP | \mathscr{D} (T \in \text{Lip}(\beta, X)^*, 1 < \beta < 2)$ concealed in the proof.

If $0 < \beta < \lambda \notin \mathbb{Z}$, and $H_p(X)$ is dense in $\text{lip}(\gamma, X)$, then $H_p(X)$ is dense in $\text{lip}(\beta, X)$. In fact, if $f \in \text{lip}(\gamma, X)$, then

$$\|f\|_{\beta,X} \leqq K_{32} \|f\|_{\gamma,X},$$

where K_{32} depends only on β, γ, d, and diam X. Hence it suffices to prove the theorem for $1 < \beta < 2$.

Fix $1 < \beta < 2$ and a compact set $X \subset \mathbb{R}^d$. Let $\alpha = \beta - 1$.

Let $T \in \text{lip}(\beta, X)^* \cap H_1(X)^\perp$. For $f \in \mathscr{E}$, and $1 \leqslant j \leqslant d$, the functions $L_j f$, defined by

$$L_j f(x,y) = \frac{D_j f(x) - D_j f(y)}{|x-y|^\alpha}\ (x, y \in \mathbb{R}^d),$$

lie in $C(\mathbb{R}^{2d})$. The map

$$f \mapsto Lf = (L_1 f, L_2 f, \ldots, L_d f)$$
$$\mathscr{E} \to C(\mathbb{R}^{2d})^d$$

has for kernel the space of affine functions on $\mathbb{R}^d$, a subspace of $H_1(X)$. Thus we obtain a well-defined functional S on imL by setting

$$SLf = Tf\ (f \in \mathscr{E}).$$

Suppose T is actually bounded with respect to the quotient seminorm $\|\cdot\|'_{\beta,X}$ *induced on* lip(β, X) *by the "pure"* Lip β *seminorm on* lip β:

$$\|f\|'_\beta=\sum_{j=1}^{d} \sup_{x\neq y} \frac{|D_jf(x)-D_jf(y)|}{|x-y|^\alpha}.$$

Then we have

$$\begin{aligned}|SLf|=|Tf|&\leqq\|T\| \ \|f\|'_{\beta,X}\\ &\leqq\|T\| \ \|f\|'_\beta=\|T\| \ \|Lf\|_{u(d)} \quad (f\in\mathcal{D}),\end{aligned}$$

where

$$\|g\|_{u(d)}=\sum_{1}^{d}\|g_j\|_u \quad (g\in C(\mathbb{R}^{2d})^d)$$

is the product uniform norm. By the Hahn-Banach Theorem, S has a continuous extension $\tilde{S}$ on $C(\mathbb{R}^{2d})^d$. We may take $\tilde{S}$ to have support in $X\times X$, since $SLf=0$ whenever $Lf=0$ on a neighbourhood of $X\times X$ (If $Lf=0$ on a neighbourhood of $X\times X$, then each D_jf is constant on that neighbourhood, hence X has another neighbourhood on each component of which f is affine, hence $Tf=0$, hence $SLf=0$.). By the Riesz Representation Theorem, there exist measures $\mu_1, \mu_2, \ldots, \mu_d$ on $X\times X$ such that

$$Tf=\sum_{1}^{d}\int L_jf\,d\mu_j \ (f\in\mathcal{E}).$$

Thus for $f\in\mathcal{D}$,

$$\begin{aligned}TPf&=\sum_{j=1}^{d}\int L_jPf\,d\mu_j\\
&=\sum_{j=1}^{d}\int\frac{D_jPf(x)-D_jPf(y)}{|x-y|^\alpha}\,d\mu_j(x,y)\\
&=\sum_{j=1}^{d}\int\frac{PD_jf(x)-PD_jf(y)}{|x-y|^\alpha}\,d\mu_j(x,y)\\
&=-\lambda\sum_{j=1}^{d}\iint\frac{D_jf(\xi)}{|x-y|^\alpha}\left\{\frac{1}{|\xi-x|^{d-2}}-\frac{1}{|\xi-y|^{d-2}}\right\}d\mathcal{L}^d(\xi)d\mu_j(x,y)\\
&=\lambda(d-2)\sum_{j=1}^{d}\iint\frac{f(\xi)}{|x-y|^\alpha}\left\{\frac{\xi_j-x_j}{|\xi-x|^{d-2}}-\frac{\xi_j-y_j}{|\xi-y|^{d-2}}\right\}d\mathcal{L}^d(\xi)d\mu_j(x,y)\\
&=\int\left[\lambda(d-2)\sum_{j=1}^{d}\int\frac{1}{|x-y|^\alpha}\left\{\frac{\xi_j-x_j}{|\xi-x|^{d-2}}-\frac{\xi_j-y_j}{|\xi-y|^{d-2}}\right\}d\mu_j(x,y)\right]f(\xi)\,d\mathcal{L}^d(\xi)\end{aligned}$$

$$=\int G(\xi)f(\xi)\,d\mathscr{L}^d\xi\,,$$

where $G(\xi)$ denotes the expression in square brackets. The Gauss-Green theorem was used to get the fifth line. The use of Fubini's Theorem to get the sixth line is justified by the fact that we may insert absolute values in the fifth line and get something bounded by $\|f\|_\infty$; in fact this is true for any function f in L^∞ with compact support. This estimate is basically the same as Lemma 2. Hence G is locally integrable with respect to $\mathscr{L}^d$.

Now suppose $\mathscr{L}^d(X)=0$. Fix $\xi\notin X$, and choose $\psi\in\mathscr{D}$ such that

$$\psi(x)=\frac{1}{|x-\xi|^{d-2}}$$

for x near X. Then $\psi\in H_1(X)$, hence

$$0=T\psi=G(\xi).$$

Thus $G=0$ off X, hence $G=0$ $\mathscr{L}^d$-almost everywhere, hence $TP\,|\,\mathscr{D}=0$, hence $T\,|\,\mathscr{E}=0$, hence $T\perp \text{lip}(\mathscr{D}, X)$.

Thus, if $\mathscr{L}^d(X)=0$, then $H_1(X)$ is dense in lip(β, X) with respect to the "pure" seminorm $\|\cdot\|'_{\beta,X}$. But this clearly implies that $H_1(X)$ is dense with respect to the usual Lip(β, X) norm, $\|\cdot\|_{\beta,X}$, since $H_1(X)$ contains all affine functions.

The theorem fails for $\beta=2$. In fact there are countable compact sets X such that $H_1(X)$ is not dense in $D^2(X)$; all that is necessary is that $J_2(X, a)$ contain $\triangle\cdot(a)$ for some $a\in X$.

References

[1] O'Farrell, A. Annihilators of rational modules. *J. funct. Anal.* (to appear).

[2] O'Farrell, A. Localness of certain Banach modules. *Indiana Univ. math. J.* (to appear).

[3] Stein, E. 1970 *Singular integrals and differentiability properties of functions.* Princeton.

[4] Weinstock, B. 1973 Uniform approximation by solutions of elliptic equations. *Proc. Am. math. Soc.* **41**, 513–517.

Added in proof

The author has recently extended Theorem 1 to the case of integral β.