LIPSCHITZ FUNCTIONS AND BAD METRICS

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We explore conditions under which a metric space admits metrics that are in a certain sense smaller than the original metric. The question is related to the existence of certain Lipschitz functions, and the answers throw light upon some problems encountered by J. D. Stein [6], concerning the Lipschitz index of a metric space and the ideal structure of Lipschitz algebras.

1. BAD METRICS

Let (X, ρ) be a metric space. We say that a pseudometric σ on X (see [4]) is *smaller* than ρ on X if for each point x in X the quotient

$$\frac{\sigma(x, y)}{\rho(x, y)} \tag{1}$$

converges to zero as $\rho(x, y) \downarrow 0$, $y \in X$. We say that σ is *much smaller* than ρ on X if for each point x in X the quotient

$$\frac{\sigma(y, z)}{\rho(y, z)}$$

converges to zero as $\rho(x, y) \downarrow 0$ and $\rho(x, z) \downarrow 0$. A metric σ on X is *larger* than ρ if for each x in X the quotient (1) tends to infinity as $\rho(x, y) \downarrow 0$. Observe that the statements "σ is larger than ρ" and "ρ is smaller than σ" are not necessarily equivalent. They are equivalent if σ induces the same topology as ρ. Also, ρ is smaller than ρ if and only if (X, ρ) is a discrete topological space. If h(r) is a concave, increasing function, defined for $r \geq 0$, with $h(0) = 0$, then $h \circ \rho$ is a metric on X. The spaces (X, ρ) and $(X, h \circ \rho)$ are homeomorphic if the restriction of h to the image of ρ is continuous at 0. Thus there exist metrics on X that are larger than ρ and induce the same topology. On the other hand, it may happen that there are no metrics, or even nonzero pseudometrics, smaller than ρ. For instance, if σ is a pseudometric on Euclidean space $\mathbb{R}^n$ that is smaller than the Euclidean metric, then

$$\frac{|\sigma(x, z) - \sigma(y, z)|}{|x - y|} \leq \frac{\sigma(x, y)}{|x - y|},$$

hence the gradient $\nabla_x \sigma(x, z)$ vanishes identically as a function of x, for each fixed z, hence σ is identically zero.

We say that ρ is a *bad metric* on X if X admits a smaller metric that induces the same topology. We say that ρ is a *good* metric if X admits no smaller metric,

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and that ρ is a *very good* metric if X admits no smaller pseudometric. Thus a very good metric is good, and a good metric is not bad. The square root of the Euclidean metric on $\mathbb{R}^n$ is bad, whereas the Euclidean metric itself is very good. If X is the union of two disjoint closed balls in $\mathbb{R}^n$, then the Euclidean metric is good on X, but not very good.

We ask: how can we tell, by looking at (X, ρ) as a metric space, whether ρ is bad, good, or very good? If ρ is bad, how can we construct a smaller metric giving the same topology? If ρ is not very good, how can we construct a smaller pseudometric? For instance, given $\mathbb{R}^n$ with the square root of the Euclidean metric, how can we "rediscover" the Euclidean metric, using only metric-space operations? (By metric-space operations we mean functors on the category of metric spaces with isometries.)

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2. LIPSCHITZ FUNCTIONS

Let (X, ρ) be a metric space. For $\beta > 0$, the space $\mathrm{Lip}(\beta, \rho)$ consists of all real-valued functions f on X for which the pseudonorm

$$\|f\|_{\beta,\rho} = \sup\left\{\frac{|f(x) - f(y)|}{\rho(x, y)^\beta}: x, y \in X,\ x \neq y\right\}$$

is finite. If $0 < \beta \leq 1$, then for each point z in X the function $x \to \rho(z, x)^\beta$ belongs to $\mathrm{Lip}(\beta, \rho)$, and hence $\mathrm{Lip}(\beta, \rho)$ separates points on X (that is, for each pair of points x and y in X, there exists a function $f \in \mathrm{Lip}(\beta, \rho)$ such that $f(x) \neq f(y)$). On the other hand, if $\beta > 1$, then it may happen that $\mathrm{Lip}(\beta, \rho)$ consists only of the constants. Indeed, $\mathrm{Lip}(\beta, \rho) \neq \mathbb{R}$ if and only if X admits a pseudometric σ such that $\sigma \leq \rho^\beta$, hence $\mathrm{Lip}(\beta, \rho) = \mathbb{R}$ where $\beta > 1$ and ρ is very good.

LEMMA 1. *Let (X, ρ) be a metric space, and let $\beta > 0$. Suppose $\mathrm{Lip}(\beta, \rho)$ separates points on X. Then there is a metric σ on X such that $\sigma \leq \rho^\beta$, $\mathrm{Lip}(\beta, \rho) = \mathrm{Lip}(1, \sigma)$, and $\|f\|_{\beta,\rho} = \|f\|_{1,\sigma}$ whenever $f: X \to \mathbb{R}$.*

Proof. We simply define σ as the Gleason metric [3] of $\mathrm{Lip}(\beta, \rho)$:

$$\sigma(x, y) = \sup\{|f(x) - f(y)|: \|f\|_{\beta,\rho} \leq 1\}.$$

It is easy to see that σ is a metric and that $\sigma \leq \rho^\beta$.

Let $f \in \mathrm{Lip}(\beta, \rho)$. Then, by definition of σ,

$$|f(x) - f(y)| \leq \|f\|_{\beta,\rho}\, \sigma(x, y)$$

whenever $x, y \in X$. Hence $f \in \mathrm{Lip}(1, \sigma)$ and $\|f\|_{1,\sigma} \leq \|f\|_{\beta,\rho}$.

Conversely, let $f \in \mathrm{Lip}(1, \sigma)$. Then

$$|f(x) - f(y)| \leq \|f\|_{1,\sigma}\, \sigma(x, y) \leq \|f\|_{1,\sigma}\, \rho(x, y)^\beta.$$

Thus $f \in \mathrm{Lip}(\beta, \rho)$ and $\|f\|_{\beta,\rho} \leq \|f\|_{1,\sigma}$.

The converse to this lemma is trivial: if there exists a metric σ on X such that $\sigma \leq \rho^\beta$, then $\mathrm{Lip}(\beta, \rho)$ separates points on X.

In order to apply the lemma to the study of bad metrics, we need to know whether σ induces the same topology as ρ.

LEMMA 2. *Let* (X, ρ), β, *and* σ *be as in Lemma* 1. *Then* ρ *and* σ *induce the same topology on* X *if and only if for each point* x *in* X *and each positive* ε *there exists a positive* η *such that, for each point* y *in* X *such that* $\rho(x, y) > \varepsilon$, *there exists a function* f *such that*

$$\|f\|_{\beta,\rho} \leq 1 \qquad \textit{and} \qquad |f(x) - f(y)| > \eta .$$

Proof. We say that a set $U \subset X$ is a *ρ-neighborhood* of the point x if U is a neighborhood of x in the topology induced by ρ. Since $\sigma \leq \rho^\beta$, it follows that for each point x in X, every σ-neighborhood of x is a ρ-neighborhood of x. Hence ρ and σ induce the same topology if and only if for every point x each ρ-neighborhood of x is a σ-neighborhood. Fix $x \in X$ and $\varepsilon > 0$. The basic ρ-neighborhood of x,

$$\{y \in X: \rho(x, y) < \varepsilon\},$$

is a σ-neighborhood of x if and only if there exists a constant $\eta > 0$ such that for each y in X,

$$\sigma(x, y) \leq \eta \qquad \text{implies} \qquad \rho(x, y) < \varepsilon . \tag{2}$$

The following three statements are clearly equivalent, for fixed x, ε, and η.

1. For all y in X, (2) holds.

2. For all y in X,

$$\rho(x, y) \geq \varepsilon \qquad \text{implies} \qquad \sigma(x, y) > \eta .$$

3. For all y in X with $\rho(x, y) \geq \varepsilon$, there exists a function f such that $\|f\|_{\beta,\rho} \leq 1$ and $|f(x) - f(y)| > \eta$.

The result follows.

Another condition that implies the two spaces are homeomorphic is that for each point x in X there exists a constant $\kappa > 0$ such that for each y in X there exists a function f such that $\|f\|_{\beta,\rho} \leq 1$ and

$$\kappa\, \rho(x, y)^\beta \leq |f(x) - f(y)| .$$

This condition is satisfied if each pair $(x, y) \in X \times X$ is a *peak pair* for $\mathrm{Lip}(\beta, \rho)$, in the sense that there exists a function f in $\mathrm{Lip}(\beta, \rho)$ such that $\|f\|_{\beta,\rho} = 1$ and $|f(x) - f(y)| = \rho(x, y)^\beta$.

PROPOSITION 1. *If* (X, ρ) *is a compact metric space,* $\beta > 1$, *and* $\mathrm{Lip}(\beta, \rho)$ *separates points, then the Gleason metric* σ *of* $\mathrm{Lip}(\beta, \rho)$ *induces the same topology as* ρ, *hence* ρ *is a bad metric on* X.

Proof. Fix $x \in X$ and $\varepsilon > 0$. For each point $y \in X$ with $\rho(x, y) \geq \varepsilon$, let $\eta_y = 2^{-1}\sigma(x, y)$. Then the set

$$N_y = \{z \in X: \text{there exists a function } f \text{ such that } \|f\|_{\beta,\rho} \leq 1 \text{ and } |f(z) - f(x)| > \eta_y\}$$

is an open neighborhood of x. Thus the family $\{N_y: \rho(x, y) \geq \varepsilon\}$ is an open covering of the set

$$T = \{y \in X: \rho(x, y) \geq \varepsilon\},$$

which is compact. Thus there exists a finite number of points $y_1, y_2, \cdots, y_m$ in X such that $\rho(x, y_i) \geq \varepsilon$ and

$$T \subset N_{y_1} \cup N_{y_2} \cup \cdots \cup N_{y_m}.$$

Let

$$\eta = \min\{\eta_{y_i}: 1 \leq i \leq m\}.$$

Then for each point y in X with $\rho(x, y) \geq \varepsilon$, there exists a function f such that $\|f\|_{\beta,\rho} \leq 1$ and $|f(x) - f(y)| > \eta$. Thus, by Lemma 2, ρ and σ induce the same topology on X.

A slight variation on this argument shows that *if* (X, ρ) *is locally compact and* $\mathrm{Lip}(\beta, \rho)$ *is regular* (that is, separates points from closed sets), *then* ρ *is a bad metric on* X.

In case (X, ρ) is compact, $\mathrm{Lip}(\beta, \rho)$ forms an algebra under pointwise addition and multiplication. Endowed with the norm

$$\|f\| = \|f\|_{\beta,\rho} + \sup_X |f|,$$

$\mathrm{Lip}(\beta, \rho)$ becomes a commutative Banach algebra with identity [5], [6]. The algebra $\mathrm{Lip}(1, \rho)$ has been studied extensively [2], [5], [7], from the point of view of ideal theory and derivations. These results extend automatically to $\mathrm{Lip}(\beta, \rho)$ for $0 < \beta < 1$. J. D. Stein [6] attempted to extend the ideal theory of D. R. Sherbert and L. Waelbroeck to $\mathrm{Lip}(\beta, \rho)$, for $\beta > 1$. The foregoing results show that every $\mathrm{Lip}(\beta, \rho)$ space is isometrically isomorphic to some space $\mathrm{Lip}(1, \tilde{\sigma})$ (where $\tilde{\sigma}$ is a metric on a quotient space of (X, ρ)), and hence the theory of Lip β is contained in the theory of Lip 1. In particular, every closed ideal in $\mathrm{Lip}(\beta, \rho)$ is the intersection of closed primary ideals (see [5], [6], [7]).

3. CONNECTEDNESS

Let (X, ρ) be a metric space. If X is the union of two subsets that lie at positive distance from one another, then ρ is not very good. This remark has a local analogue.

PROPOSITION 2. *Let* (X, ρ) *be a locally compact, totally disconnected, metric space. Then* ρ *is a bad metric on* X.

Proof. Let E be a closed subset of X, and let p be a point of $X \sim E$. Then there exist a compact neighborhood Y of p and a closed neighborhood Z of E such that

$$X = Y \cup Z, \quad Y \cap Z = \emptyset.$$

Since Y is compact, the distance between Y and Z is positive, hence the characteristic function of Y belongs to $\mathrm{Lip}(2, \rho)$. Thus $\mathrm{Lip}(2, \rho)$ is regular, and the proposition follows from the remark after Proposition 1.

In case X is locally compact, the next proposition is a special case of the last.

PROPOSITION 3. *Let* (x, ρ) *be a metric space. Suppose there exist a positive constant* κ *and a sequence of positive numbers* $\varepsilon_n \downarrow 0$ *such that, for each* n, X *admits a covering* $\{U_\alpha^n\}_\alpha$ *by pairwise disjoint sets such that* $\operatorname{diam} U_\alpha^n \leq \varepsilon_n$ *and*

$$\operatorname{dist}[U_\alpha^n, U_\beta^n] \geq \kappa\,\varepsilon_n \quad (\alpha \neq \beta).$$

Then ρ *is bad.*

Proof. Choose a sequence of positive numbers $\eta_n \downarrow 0$ such that

$$\sum_{n=m}^{\infty} \eta_n < \varepsilon_m^2 \quad (m = 1, 2, 3, \cdots).$$

Let χ_α^n denote the characteristic function of U_α^n, and define

$$\sigma(x, y) = \sum_{n=1}^{\infty} \sum_{\alpha} \eta_n \left| \chi_\alpha^n(x) - \chi_\alpha^n(y) \right|,$$

whenever $x, y \in X$. Clearly, σ is a metric on X. If $\kappa\,\varepsilon_{m+1} \leq \rho(x, y) < \kappa\,\varepsilon_m$, then for each $n \leq m$, x and y belong to the same U_α^n. Thus

$$\sigma(x, y) \leq 2 \sum_{n=m+1}^{\infty} \eta_n < 2\varepsilon_{m+1}^2 < 2\kappa^{-2}\rho(x, y)^2.$$

Hence $\sigma(x, y) \leq 2\kappa^{-2}\rho(x, y)^2$ whenever $\rho(x, y) \leq \kappa\,\varepsilon_1$, so that σ is much smaller than ρ. In order to show that σ induces the same topology as ρ, it suffices to show that for each point x in X, each ρ-neighborhood of x is a σ-neighborhood, that is, that we can force $\rho(x, y)$ to be small by making $\sigma(x, y)$ small. However, if $\sigma(x, y) < \eta_m$, then x and y belong to the same U_α^m, and hence $\rho(x, y) \leq \varepsilon_m$. Hence σ does induce the same topology as ρ, so that ρ is a bad metric on X.

A *net* on the unit interval I is a finite ordered set of points

$$0 = a_0 < a_1 < \cdots < a_n = 1.$$

Let Γ be an arc (Γ is a *set*) in the metric space (X, ρ), and let $\phi\colon I \to \Gamma$ be a homeomorphism. A *net* on Γ is an ordered set $(b_0, \cdots, b_n)$ such that $(\phi^{-1}(b_0), \cdots, \phi^{-1}(b_n))$ is a net on I. The ρ-*length* of Γ is the supremum over all nets on Γ of the sums

$$\sum_{i=1}^{n} \rho(b_i, b_{i-1}).$$

If the ρ-length of Γ is finite, we say Γ is ρ-*rectifiable*. If Γ is a countable union of ρ-rectifiable subarcs, then we say that Γ is *countably* ρ-*rectifiable*.

PROPOSITION 4. *Let* (X, ρ) *be a metric space.*

(i) *If* X *contains a* ρ-*rectifiable arc, then* X *does not admit a metric much smaller than* ρ.

(ii) *If* X *is connected by countably* ρ*-rectifiable arcs, then* X *does not admit a nonzero pseudometric much smaller than* ρ.

Proof. (i) Let Γ be a ρ-rectifiable arc, with ρ-length L. Suppose, contrary to the assertion, that there exists a metric σ on X that is much smaller than ρ. Let $\varepsilon > 0$. The function F, defined on $\Gamma \times \Gamma$ by

$$F(x, y) = \begin{cases} 0 & (x = y), \\ \dfrac{\sigma(x, y)}{\rho(x, y)} & (x \neq y) \end{cases}$$

is continuous on $\Gamma \times \Gamma$ and vanishes on the diagonal D. For $\alpha > 0$, let

$$N_\alpha = \{(x, y) \in \Gamma \times \Gamma: \rho(x, y) \leq \alpha\}.$$

Then each N_α is compact, and $\bigcap_{\alpha > 0} N_\alpha = D$. Hence there exists $\delta > 0$ such that

$$N_\delta \subset \{(x, y) \in \Gamma \times \Gamma: F(x, y) \leq \varepsilon\}.$$

Thus $\rho(x, y) \leq \delta$ implies $\sigma(x, y) \leq \varepsilon \rho(x, y)$.

Choose a net $(b_0, \cdots, b_n)$ on Γ such that $\rho(b_i, b_{i-1}) \leq \delta$. Then

$$\sigma(b_0, b_n) \leq \sum_{i=1}^{n} \sigma(b_i, b_{i-1}) \leq \varepsilon L.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\sigma(b_0, b_n) = 0$, which is impossible.

Assertion (ii) is proved in a similar way.

An arc Γ is called a ρ-Lip arc if there exist a parametrization $\phi: I \to \Gamma$ and a constant $\kappa > 0$ such that

$$\rho[\phi(t), \phi(u)] \leq \kappa |t - u|, \tag{3}$$

whenever $t, u \in I$. Every ρ-Lip arc is ρ-rectifiable. A ρ-rectifiable arc is a ρ-Lip arc if and only if the parametrization induced by arc length satisfies (3).

If X *contains a* ρ-Lip *arc* Γ, *then the metric* ρ *is good;* for if σ were a smaller metric on X, then the pullback of σ from Γ to I would be a smaller metric than the pullback of ρ, and hence smaller than the Euclidean metric.

Similarly, *if* X *is connected by piecewise* ρ-Lip *arcs, then* ρ *is a very good metric.*

Stein [6] introduced the following equivalence relation on X: $x \approx y$ if and only if x and y may be joined by a ρ-Lip arc. He called the equivalence classes L-*components* of (X, ρ). He defined the *Lipschitz index* of (X, ρ) as

$$\sup\{\beta > 0: \mathrm{Lip}(\beta, \rho) \neq \mathbb{R}\},$$

and he asked whether an arcwise-connected metric space with Lipschitz index 1 need be an L-component. In view of the remark before Lemma 1, we see that the Lipschitz index of (X, ρ) is equal to the supremum of the numbers $\beta > 0$ such that

X admits a pseudometric σ satisfying the inequality $\sigma \le \rho^\beta$. Consider the unit interval I with the metric

$$\rho(t, u) = |t - u| \log \frac{4}{|t - u|} .$$

The Lipschitz index of (I, ρ) is 1, whereas I is not even countably ρ-rectifiable. Therefore the answer to Stein's question is negative. A more difficult question is whether an arcwise connected space with a very good metric ρ need be connected by countably ρ-rectifiable arcs. I do not know.

If Y is a subset of X, and ρ is a metric on X that is good on Y, then ρ is good on X. Hence, the completion of a good metric is good. However, the completion of a metric that is not good may also be good; indeed, the completion of a bad metric may be very good. For example, let ρ denote the Euclidean metric on I, and let ϕ be the Lebesgue singular function; then ϕ is a homeomorphism of I to itself, and $\phi' = 0$ on a set $E \subset I$ with full measure. Define the metric σ by setting

$$\sigma(t, u) = |\phi(t) - \phi(u)| .$$

Then σ induces the Euclidean topology, and σ is smaller than ρ on E, hence ρ is a bad metric on E. Since E is dense, the completion of (E, ρ) is (I, ρ). This example also shows that very good metrics are by no means unique: σ is also a very good metric on I.

By the method of Proposition 4, we see that if X contains an arc with ρ-Hausdorff dimension equal to β, and $\gamma > \beta$, then X does not admit a nonzero pseudometric $\sigma \le \rho^\gamma$, hence the Lipschitz index of X does not exceed β.

4. LITTLE LIP CRITERIA

Let (X, ρ) be a metric space. We define $\mathrm{lip}(\rho)$ as the space of all functions $f: X \to \mathbb{R}$ such that, corresponding to each $x \in X$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(y) - f(z)| \le \varepsilon \rho(y, z)$$

whenever $\rho(x, y) < \delta$ and $\rho(x, z) < \delta$. We define $\mathrm{flip}(\rho)$ (f for *feeble*) as the space of all functions $f: X \to \mathbb{R}$ such that, for each $x \in X$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \le \varepsilon \rho(x, y)$$

wherever $\rho(x, y) < \delta$. Clearly, $\mathrm{lip}(\rho) \subset \mathrm{flip}(\rho)$. In case (X, ρ) is compact, the number δ in the definition of $\mathrm{lip}(\rho)$ may be chosen to depend only on ε. It is easy to see that $\mathrm{flip}(\rho) = \mathbb{R}$ if and only if ρ is a very good metric on X. Also, $\mathrm{lip}(\rho) = \mathbb{R}$ if and only if X admits no nonzero pseudometric much smaller than ρ.

PROPOSITION 5. *Let* (X, ρ) *be a separable metric space.*

(i) X *admits a metric smaller than* ρ *if and only if* $\mathrm{flip}(\rho)$ *separates points on* X.

(ii) X *admits a metric much smaller than* ρ *if and only if* $\mathrm{lip}(\rho)$ *separates points on* X.

Proof. The two parts are similar in proof. We prove (i). The "only if" part is trivial.

Suppose flip(ρ) separates points on X. For each pair $(x, y) \in X \times X$ with $x \neq y$, choose a function $f \in \text{flip}(\rho)$ such that $f(x) \neq f(y)$. Define $F(x, y; z, w) = f(z) - f(w)$. The set

$$N(x, y) = \{(z, w) \in X \times X: F(x, y; z, w) \neq 0\}$$

is an open neighborhood of (x, y) in $X \times X$, and the family

$$\{N(x, y): x \neq y\}$$

is an open covering of the complement of the diagonal in the separable space $X \times X$. By Lindelöf's theorem, we may choose a countable subcover. Let $\{f_m\}_1^\infty$ be the sequence of functions in flip(ρ) corresponding to the elements of this subcover. We define

$$\sigma(x, y) = \sum_{m=1}^{\infty} 2^{-m} \|f_m\|_{1,\rho}^{-1} |f_m(x) - f_m(y)|$$

whenever $(x, y) \in X \times X$. Since $\{f_m\}_1^\infty$ separates points, it follows that σ is a metric. Let $x_n, x \in X$, with $\rho(x_n, x) \downarrow 0$. Then

$$\frac{\sigma(x, x_n)}{\rho(x, x_n)} = \sum_{m=1}^{\infty} 2^{-m} \|f_m\|_{1,\rho}^{-1} \frac{|f_m(x) - f_m(x_n)|}{\rho(x, x_n)}$$

converges to zero, by the dominated-convergence theorem. Thus σ is smaller than ρ, and the proof is complete.

If (X, ρ) is a *compact* metric space, then arguing as in Proposition 1 we see that ρ is a bad metric on X if and only if flip(ρ) separates points.

5. POOR SETS

Let (X, ρ) be a metric space, and let E be a subset of X. We say that E is a *ρ-poor* set if there exists a metric σ on X such that for each point $p \in E$ the quotient

$$\frac{\sigma(p, x)}{\rho(p, x)}$$

tends to zero as $\rho(p, x) \downarrow 0$, $x \in X$ (this is stronger than the statement that σ is smaller than ρ on E). If ρ is not a good metric on X, then every subset of X is ρ-poor. The example of Section 3, involving the Lebesgue singular function, exhibits a dense ρ-poor set, with ρ very good. The main result of this section is that every metric space contains poor sets, in fact, all reasonably small sets are poor.

The *derived set* E' of a subset E of a metric space is the set of accumulation points of E. The *second derived set* $E^{(2)}$ is E'', and so on.

PROPOSITION 6. *Let* (X, ρ) *be a metric space, and let* E *be a countable subset of* E. *Suppose there exists a positive integer* n *such that* $E^{(n)} = \emptyset$. *Then* E *is* ρ-*poor.*

First we prove the result in case $E = \{p\}$, a singleton.

LEMMA 3. *Let* (X, ρ) *be a metric space, and let* $p \in X$. *Then there exists a metric* σ *on* X *such that*

$$\sigma(p, x) \leq \rho(p, x)^2$$

whenever $x \in X$ *and* $\rho(p, x) \leq 1$.

Proof. First, suppose X is a Euclidean space, ρ is the Euclidean metric, and p is the origin. Define σ by

$$\sigma(x, y) = \rho[|x|^3 x, |y|^3 y] \tag{4}$$

for $x, y \in X$. It is clear that σ is a metric, and that

$$\sigma(0, x) = \rho(0, x)^4 \quad (x \in X).$$

Next, suppose (X, ρ) has the Euclidean four-point property (this means that each four-point subspace of X may be imbedded isometrically in $\mathbb{R}^3$). Define σ as follows. For x and y in X, set

$$r = \rho(p, x), \quad s = \rho(p, y), \quad d = \rho(x, y),$$

$$\sigma(x, y)^2 = r^8 + s^8 - r^3 s^5 - r^5 s^3 + r^3 s^3 d^2. \tag{5}$$

Observe that (5) defines the same function as (4) in case X is a Euclidean space and $p = 0$. Also, the assertion that σ is a metric involves at most four points of X at a time. For instance, when the triangle inequality $\sigma(x, y) \leq \sigma(x, z) + \sigma(y, z)$ for σ is written in terms of ρ, it involves only the ρ-distances between the points p, x, y, and z. That σ is a metric therefore follows from the Euclidean four-point property and the fact that (4) defines a metric on Euclidean space. Clearly,

$$\sigma(p, x) = \rho(p, x)^4 \quad (x \in X).$$

Finally, let (X, ρ) be an arbitrary metric space. Then [1, p. 131] the metric space $(X, \rho^{1/2})$ has the Euclidean four-point property. By the previous case, there is a metric σ on X such that

$$\sigma(p, x) = \rho(p, x)^2 \quad (x \in X),$$

and the proof is complete.

It is a simple matter, given a positive number δ, to modify the construction above so that it yields a metric σ such that

$$\sigma(p, x) = \rho(p, x)^2$$

for small $\rho(p, x)$, while

$$\sigma(x, y) = \rho(x, y)$$

whenever $\rho(p, x) \geq \delta$ and $\rho(p, y) \geq \delta$. Hence, given any countable discrete set F, we may construct a metric σ on X such that

$$\sigma(p, x) = \rho(p, x)^2$$

when $p \in F$ and $\rho(p, x)$ is small.

Now suppose E is a countable subset of X and $E^{(n)} = \emptyset$ for some positive integer n. Then $E^{(n-1)}$ is countable and discrete, and therefore we may construct a metric σ_1 on X such that

$$\sigma_1(p, x) = \rho(p, x)^2$$

for $p \in E^{(n-1)}$ and small $\rho(p, x)$, while for each $q \in X \sim E^{(n-1)}$ the quotient

$$\frac{\sigma_1(q, x)}{\rho(q, x)}$$

is bounded above and below for small nonzero $\rho(q, x)$. Next, the set $E^{(n-2)} \sim E^{(n-1)}$ is countable and discrete, and $E^{(n-1)}$ is closed, and therefore we may construct a metric σ_2 on X such that

$$\sigma_2(p, x) = \sigma_1(p, x)^2$$

for each point $p \in E^{(n-2)} \sim E^{(n-1)}$ and small $\rho(p, x)$, whereas for each point $q \notin E^{(n-2)} \sim E^{(n-1)}$ the quotient

$$\frac{\sigma_2(q, x)}{\sigma_1(q, x)}$$

is bounded above and below for small $\rho(q, x)$. Continuing in this way, we obtain a metric σ_n on X such that for each point p in the closure of E there exists a constant $\kappa > 0$ such that

$$\sigma_n(p, x) \leq \kappa\, \rho(p, x)^2$$

provided $\rho(p, x)$ is small. Thus E is ρ-poor. This concludes the proof of Proposition 6.

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