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EQUICONVERGENCE OF DERIVATIONS

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This paper is a study of bounded point derivations on the classical Banach algebras of analytic functions of a complex variable. The results are positive in character. The higher-order Gleason metrics d^p of $R(X)$ are introduced and conditions are studied under which convergence takes place with respect to these metrics. In particular, if $R(X)$ admits a pth-order bounded point derivation at a point $x \in \partial X$ and $\mathring{X}$ satisfies a cone condition at x, then $d^p(y, x)$ tends to 0 as y tends to x along the midline of the cone. Similar results hold for the other classical function algebras. In the case of the algebra $H^\infty(U)$, for open $U \subset C$, the analogous results hold only for regular derivations (a regular pth-order derivation maps z^p to a nonzero complex number). The points of the maximal ideal space of $H^\infty(U)$ at which regular bounded point derivations exist are characterized in terms of analytic capacity, following Hallstrom.

1. Let x be a point of the plane C and A be a class of functions analytic in a disc D centered at x, each function having modulus bounded by 1. Then, as is clear from Cauchy's integral formula, the family $\{f' \mid f \in A\}$ is equicontinuous at x, and for every sequence $\{x_n\} \to x$, the sequence $\{f'(x_n)\}$ converges to $f'(x)$, *uniformly on* A, i.e., $\{f'(x_n)\}$ is *equiconvergent* to $f'(x)$. More generally, for any integer $p \geqq 1$, $\{f^{(p)}(x_n)\}$ is equiconvergent to $f^{(p)}(x)$.

Now, given a C-algebra A of continuous functions on a compact set $X \subset C$ which are analytic on $\mathring{X}$, it is often possible to find points on ∂X at which nonzero point derivations exist on A. A (*first order*) *point derivation* at $x \in X$ on A is a linear functional $D: A \to C$ such that

$$D(fg) = f(x)Dg + g(x)Df \ ,$$

whenever $f, g \in A$. This notion generalizes that of derivative at a point. For points $y \in \mathring{X}$ all point derivations are of the form $f \to \alpha f'(y)$ for some complex constant α (independent of f) provided A contains the polynomials. Suppose A contains the identity map z and D is a *normalized* point derivation at x on A, i.e., $Dz = 1$. A natural question is:

Q1. *When is there a sequence of points $x_n \in \mathring{X}$, converging to x, such that the sequence $\{f'(x_n)\}$ converges to Df for all $f \in A$?*

A *bounded* point derivation is a point derivation that is continuous with respect to the uniform norm on X. If A admits a bounded point derivation D at a point x we may ask:

Q2. *Can we find $x_n \to x$, $x_n \in \mathring{X}$, such that $f'(x_n)$ is equiconvergent to Df on $A_1 = A \cap \{f \mid \|f\|_X \leqq 1\}$?*

We shall concern ourselves with Q2, which lends itself to treatment by Banach algebra techniques.

2. We treat first the case $A = R(X)$, the *uniform* closure on X of $R_0(X)$, the class of rational functions with poles off X. $R(X)$ is a *function algebra* on X [2, p. 2]. The *Gleason metric d^0 on X*, with respect to $R(X)$, is defined by

$$d^0(x, y) = \sup \{|f(x) - f(y)| \mid f \in R_0(X), \|f\|_X \leqq 1\},$$

for $x, y \in X$. Here $\|f\|_X$ denotes the sup norm of F on X. The properties of X with respect to this metric have been thoroughly investigated. An account may be found in [2], [4]. If x and y belong to the same component of $\mathring{X}$, then $d^0(x, y) < 2$. If x is a peak point for $R(X)$, then $d^0(x, y) = 2$ whenever $y \neq x$. This prompted the definition of Gleason part. A *part* P of the algebra $R(X)$ is a subset of X which forms an equivalence class under the relation $x \sim y \langle = \rangle d^0(x, y) < 2$. The structure of parts can be very complicated. Davie has shown that P may be disconnected, and the Swiss cheese example shows that P may have no interior (cf. [4]). However, a nontrivial part (a part which does not just consist of one peak point) has full area density at each of its points, and in fact Browder [2, p. 177] has shown that every Gleason ball $\{x \in X \mid d^0(x, a) < \varepsilon\}$ $(\varepsilon > 0)$ about a nonpeak point a has full area density at a.

In particular, a is not isolated in the part metric d^0, and there is a sequence of points $x_n \in P \backslash \{a\}$ which converges to a *simultaneously* in the Euclidean and Gleason metrics. In plain language, as $n \to +\infty$, $|x_n - a| \to 0$, and $\{f(x_n)\}$ is *equiconvergent* to $f(a)$ for $f \in R_0(X) \cap \{f \mid \|f\|_X \leqq 1\} = R_0(X, 1)$.

For $p \geqq 1$ we define the pth *order Gleason metric* on X by

$$d^p(x, y) = \sup \{|f^{(p)}(x) - f^{(p)}(y)| \mid f \in R_0(X, 1)\},$$

for $x, y \in X$.

The first thing to note is that $d^p(x, y)$ may be $+\infty$, so we are using the word "metric" a little loosely. An ordinary metric may be obtained from d^p by composing it with the arctangent function, but we would rather not do this. We extend d^p to $C \times C$ by writing

$d^p(x, y) = d^p(y, x) = +\infty$ whenever one of the elements x, y fails to be in X.

For $p \geqq 0$ we say that a (normalized) pth order bounded point derivation on $R(X)$ exists at a point $x \in X$ if and only if the functional $f \to f^{(p)}(x)$ on $R_0(X)$ extends to a continuous linear functional D_x^p on $R(X)$, i.e., if and only if

$$s^p(x) = \sup\{|f^{(p)}(x)| \mid f \in R_0(X, 1)\} = \|D_x^p\|$$

is finite. Suppose this happens, and x_n is a sequence of points of $\mathring{X}$ tending to x (in Euclidean norm). Then to say that $f^{(p)}(x_n) \to D_x^p f$ equiconvergently on $R(X, 1)$ is the same thing as saying that $d^p(x_n, x) \to 0$.

Notice that the two definitions so far available for a normalized first order bounded point derivation on $R(X)$ agree.

For purposes of computation it is usually easier to work with the function d_0^p, defined by

$$d_0^p(x, y) = \sup\{|f^{(p)}(y)| \mid f \in R_0(X, 1) \text{ and } f(x) = f'(x) = \cdots = f^{(p)}(x) = 0\} .$$

3. The elementary properties of the functions d^p, s^p, d_0^p are summarized in the following theorem. Here, as usual, p is a non-negative integer.

THEOREM 1. *Let $x, y \in C$. Then*

(1) $|s^p(x) - s^p(y)| \leqq d^p(x, y) \leqq s^p(x) + s^p(y)$;

(2) $d^p(x, y) \geqq (p + 1)!\,|x - y|/(\operatorname{diam} X)^{p+1}$;

(3) *for* $x \in \mathring{X}$,

$$s^{p+1}(x) = \lim_{y \to x} \frac{d^p(x, y)}{|x - y|} ;$$

(4) *for each compact subset K of a component of $\mathring{X}$ there is a constant $L > 0$ such that*

$$d^p(x, y) \leqq L\,|x - y| ,$$

for $x, y \in K$, so d^p is continuous on $\mathring{X}$;

(5) *s^p is continuous on $\mathring{X}$;*

(6) $d_0^p(x, y) \leqq d^p(x, y) \leqq \{1 + \exp(\operatorname{diam} X)\}\{\sup_{0 \leqq \nu \leqq p} s^\nu(x)\} d_0^p(x, y)$;

(7) *if X_n is a decreasing sequence of compact sets, each containing X in its interior, whose intersection is X, then $s_n^p \uparrow s^p$ and $d_n^p \uparrow d^p$, where s_n^p and d_n^p are respectively, the s^p-function and the d^p-function associated with X_n;*

(8) *s^p and d^p are lower semi-continuous;*

(9) *if $|x_n - x| \to 0$ and $\{s^p(x_n)\}$ is a bounded sequence, then $s^p(x)$ is finite;*

(10) *if $s^p(w) < +\infty$ for some $w \neq x$, then $s_p(x) = +\infty$ if and only if $d^p(x, y) = +\infty$ for every $y \neq x$;*

(11) *x is an interior point of X if and only if*

$$\sup_{n\geq 1}\left[\frac{1}{n}\log\frac{s_n(x)}{n!}\right] < +\infty .$$

Proof.

(1) is clear.

(2): Take $f(z) = (z-y)^{p+1}/(\text{diam } X)^{p+1}$. Then $f \in R_0(X, 1)$, so

$$d^p(x, y) \geqq |f^{(p)}(x) - f^{(p)}(y)| .$$

(3) requires a lengthy but straightforward argument, using the Cauchy integral formula.

(4) follows from (3), using compactness.

(5) follows from (1) and (4).

(6): For the second inequality, let $f \in R_0(X, 1)$, and form

$$g(z) = f(z) - \sum_{\nu=0}^{p}\frac{f^{(\nu)}(x)}{\nu!}(z-x)^\nu .$$

Then $g(x) = g'(x) = \cdots = g^{(p)}(x) = 0$, and

$$\begin{aligned}\|g\|_X &\leqq 1 + \sum_{\nu=0}^{p}\frac{s^\nu(x)}{\nu!}(\text{diam } X)^\nu \\ &\leqq \left\{1 + \sum_{\nu=0}^{p}\frac{(\text{diam } X)^\nu}{\nu!}\right\}\left\{\sup_{0\leqq\nu\leqq p} s^\nu(x)\right\} \\ &\leqq \{1 + \exp(\text{diam } X)\}\left\{\sup_{0\leqq\nu\leqq p} s^\nu(x)\right\} .\end{aligned}$$

(7) follows from the fact that each $f \in R_0(X, 1)$ belongs to every $R(X_n)$ from some point on.

(8): By (4), (5), and (7), s^p and d^p are increasing limits of continuous functions.

(9): Take $X_m \downarrow X$ as in (7). For each m, $x \in \mathring{X}_m$, so by (5),

$$\begin{aligned}s_m^p(x) &\leqq \sup_{n\geq 1} s_m^p(x_n) \\ &\leqq \sup_{n\geq 1} s^p(x_n) .\end{aligned}$$

Thus, by (7),

$$s^p(x) = \lim_{m\to\infty} s_m^p(x) \leqq \sup_{n\geq 1} s^p(x_n) < +\infty .$$

(10): We may assume $p > 0$. If $d^p(x, y) = +\infty$ for every $y \neq x$, then by (1),

$$s^p(x) \geqq d^p(x, w) - s^p(w) = +\infty .$$

This proves one direction.

If $s^p(x) = +\infty$ and $d^p(x, y) < +\infty$ for some y, then assume p is minimal. We have $x \in X$ and so we may choose a sequence $f_n \in R_0(X, 1)$ such that

$$| f_n^{(p)}(x) | \longrightarrow +\infty ,$$

while $| f_n^{(p)}(x) - f_n^{(p)}(y) | \leqq M$ for all n, for some constant M. Form

$$g_n(z) = (2z - x - y) f_n(z) .$$

Then

$$g_n^{(p)}(z) = 2p f_n^{(p-1)}(z) + (2z - x - y) f_n^{(p)}(z) .$$

Thus

$$\begin{aligned} &| g_n^{(p)}(x) - g_n^{(p)}(y) | \\ &\quad = | 2p f_n^{(p-1)}(x) + (x - y) f_n^{(p)}(x) - 2p f_n^{(p-1)}(y) - (y - x) f_n^{(p)}(y) | \\ &\quad \geqq | x - y | \, | f_n^{(p)}(x) + f_n^{(p)}(y) | \\ &\qquad - 2p \, | f_n^{(p-1)}(x) - f_n^{(p-1)}(y) | \longrightarrow +\infty \text{ as } n \longrightarrow +\infty . \end{aligned}$$

(11): The point x is an interior point of x if and only if

$$s_n(x) \leqq M^n n \, !$$

for some constant $M > 0$. ("Only if" is clear, and "if" is true because the inequality implies that every function in $R_0(X, 1)$ is actually analytic in a full disc centered at x. This forces $x \in X$.) (11) is just a way of rewriting this.

4. For our purposes all measures will be finite complex Borel regular measures with compact support in C. For $\nu > 0$, the *potential of order* ν of a measure μ is given by

$$\mu^\nu(z) = \int \frac{d|\mu|(\zeta)}{|\zeta - z|^\nu} ,$$

where $|\mu|$ is the total variation measure of μ. Wherever $\mu^1(z) < +\infty$ we define the *Cauchy transform* of μ by

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z} .$$

For every continuous linear functional L on $R(X)$ there is a measure

μ, supported on X, which "represents L on $R(X)$", i.e.,

$$\int f d\mu = Lf$$

for every $f \in R(X)$. This fact follows from the Hahn-Banach and Riesz Representation theorems. Also, μ may be chosen to have its support on ∂X, since $R(X)$ and $R(X) \mid \partial X$ are isomorphic Banach algebras. An annihilating measure for $R(X)$ is a measure μ on X such that

$$\int f d\mu = 0$$

for every $f \in R(X)$. We write $\mu \perp R(X)$. The following easy fact was first noted by Bishop, and plays a central role in our theory (cf. [2, p. 171]).

LEMMA. *If $\mu \perp R(X)$, $\mu^1(y) < +\infty$, and $\hat{\mu}(y) \neq 0$, then the measure*

$$\frac{1}{\hat{\mu}(y)} \frac{1}{z - y} \mu$$

represents "evaluation at y" on $R(X)$, i.e.,

$$\int f d\mu = f(y)$$

for $f \in R(X)$.

The case $p = 0$ of the following theorem is due to Browder [2, p. 176].

THEOREM 2. *Let p be a nonnegative integer. Suppose the measure μ represents a bounded* p*th order point derivation on $R(X)$ at x. Then for every given $a > 0$ there is a corresponding $b > 0$ such that $d^p(x, y) < a$ whenever*

$$\sum_{\nu=1}^{p+1} | x - y |^{\nu} \mu^{\nu}(y) < b \,. \tag{2}$$

Proof. We proceed by induction on p: Suppose p is the least nonnegative integer for which the proposition fails. Let μ represent D_x^p and $a > 0$ be given. We may suppose $a < 1$. For $\tau = 0, 1, \cdots, p - 1$, $R(X)$ admits a bounded τth order point derivation at x, represented by

$$\mu_\tau = \frac{\tau!(z-x)^{p-\tau}}{p!}\,\mu\,,$$

so there are numbers $b_\nu > 0$ such that

$$\sum_{\nu=0}^{\tau+1} |x-y|^\nu \mu_\tau^\nu(y) < b_\tau \tag{3}$$

forces $d^\tau(x, y) < a/2$. Now

$$\mu_\tau^\nu(y) = \int \frac{\tau!\,|z-x|^{p-\tau}}{|z-y|^\nu}\, d\,|\mu|\,(z)$$
$$\leqq \tau!\,(\operatorname{diam} X)^{p-\tau} \mu^\nu(y)\,,$$

so, setting $c_\tau = b_\tau\{\sup_{0\leqq\tau\leqq p} \tau!\,(\operatorname{diam} X)^{p-\tau}\}^{-1}$, and $c = \inf_{0\leqq\tau\leqq p-1} c_\tau$, we deduce that $\sum_{\nu=0}^{p+1} |x-y|^\nu \mu^\nu(y) < c$ forces (3) for $\tau = 0, 1, \cdots, p-1$.

Let $K = 1 + \exp(\operatorname{diam} X)$,

$$T = 2\{\sup_{0\leqq\tau\leqq p} s^\nu(x)\}K\,.$$

Note that $T \geqq 2K$, since $s^0(x) = 1$.

Choose $b > 0$ to be smaller than each of the numbers c, $1/2$, $p!\,(\operatorname{diam} X)^{-p-1}$ and $a\{2T(K_p + \|\mu\|)\}^{-1}$, where $K_p > 0$ is a constant, depending only on p, which will be described later.

Let (2) hold. We will show that $d^p(x, y) < a$. We claim it suffices to show

$$d_0^p(y, x) < a/T\,. \tag{4}$$

For, assuming (4), we have by Theorem 1(6), (1),

$$d^p(x, y) \leqq K\{\sup_{0\leqq\nu\leqq p} s^\nu(y)\} d_0^p(y, x)$$
$$\leqq K\{\sup_{0\leqq\nu\leqq p} s^\nu(x) + \sup_{0\leqq\nu\leqq p} d^\nu(x, y)\} d_0^p(y, x)\,.$$

Thus, if $d^p(x, y) \geqq a$, then $d^p(x, y) = \sup_{0\leqq\nu\leqq p} d^\nu(x, y)$, since (3) holds for $\tau = 0, 1, \cdots, p-1$, so

$$d^p(x, y)\{1 - K d_0^p(y, x)\} \leqq \frac{1}{2} T d_0^p(y, x)\,.$$

Since $K d_0^p(y, x) < aK/T < a/2 < 1/2$, we deduce

$$d^p(x, y) \leqq T d_0^p(y, x) < a\,,$$

which is a contradiction.

We proceed to get (4).

The measure $\mu_0 = ((z-x)^p/p!)\mu$ represents evaluation at x on $R(X)$. Thus $\sigma = (z-x)\mu_0$ annihilates $R(X)$. Now

$$\hat{\sigma}(y) = \frac{1}{p!}\int \frac{(z-x)^{p+1}}{z-y}\,d\mu(z) = 1 + (y-x)\hat{\mu}_0(y)\ ,$$

so, since

$$\begin{aligned}|(y-x)\hat{\mu}_0(y)| &\leqq |y-x|\,\mu_0^1(y)\\ &\leqq \frac{|y-x|\,(\operatorname{diam} X)^{p+1}\mu^1(y)}{p!} < b < 1\ ,\end{aligned}$$

we have $\hat{\sigma}(y) \neq 0$. Also $\sigma^1(y) < +\infty$, since $\mu^1(y) < +\infty$, by (2). Thus, by the lemma, the measure

$$\frac{\sigma}{\hat{\sigma}(y)(z-y)} = \frac{(z-x)^{p+1}\mu}{p!\hat{\sigma}(y)(z-y)}$$

represents evaluation at y on $R(X)$, so

$$\frac{(z-x)^{p+1}\mu}{\hat{\sigma}(y)(z-y)^{p+1}}$$

annihilates the class

$$B = \{f \in R_0(X, 1)\,|\, f(y) = f'(y) = \cdots = f^{(p)}(y) = 0\}\ ,$$

since $\mu^{p+1}(y) < +\infty$, by (2).

Let $e = \hat{\sigma}(y)$. Then $|e| > 1 - b > 1/2$, and also $|1 - e| < b$.

We have

$$\begin{aligned}d_0^p(y, x) &= \sup\{|f^{(p)}(x)|\ |\ f \in B\}\\ &= \sup\left\{\left|\int f(z)\left\{1 - \frac{(z-x)^{p+1}}{e(z-y)^{p+1}}\right\}d\mu(z)\right|\ \middle|\ f \in B\right\}\\ &\leqq \int\left|\frac{e(z-y)^{p+1} - (z-x)^{p+1}}{e(z-y)^{p+1}}\right| d\,|\mu|\,(z)\\ &\leqq \frac{1}{1-b}\int\left|\frac{(z-y)^{p+1} - (z-x)^{p+1}}{(z-y)^{p+1}} - (1-e)\right| d\,|\mu|\,(z)\\ &\leqq 2\,|x-y|\int\left|\sum_{\nu=0}^{p}\binom{p}{\nu}\frac{(z-x)^{\nu}}{(z-y)^{\nu+1}}\right| d\,|\mu|\,(z) + 2b\,\|\mu\|\ .\end{aligned}$$

Now we observe that $(z-x)^{\nu}/(z-y)^{\nu+1}$ is a linear combination of terms

$$\frac{1}{z-y}\,,\ \frac{x-y}{(z-y)^2}\,,\ \cdots\,,\ \frac{(x-y)^{\nu}}{(z-y)^{\nu+1}}\ ,$$

so that we may continue the inequality:

$$\leqq 2K_p\,|x-y|\sum_{\nu=1}^{p+1}|x-y|^{\nu-1}\mu^{\nu}(y) + 2b\,\|\mu\|\ ,$$

where K_p depends only on p, and so, continuing:

$$\leqq 2(K_p + \| \mu \|)b$$
$$\leqq \frac{a}{T}.$$

This concludes the proof.

5. We now establish a convergence theorem for the d^p metric.

THEOREM 3. *Suppose $p = 0$, and x is not a peak point for $R(X)$, or $p \geqq 1$, and $R(X)$ admits a bounded* pth *order point derivation at x. Suppose there is a positive constant K, and a sequence of points $\{y_n\}$, elements of $\mathring{X}$, which converges to x (in Euclidean norm), such that*

$$\text{dist}\,[y_n, \partial X] \geqq K \,|\, y_n - x \,| \tag{3}$$

for $n = 1, 2, 3, \cdots$. Then $\{y_n\}$ converges to x in the d^p metric.

Proof. Select a measure μ, supported on ∂X, with no mass at x, which represents the pth order derivation at x.

By Theorem 2, it suffices to show that $|x - y_n|^\nu \mu^\nu(y_n)$ is small for each ν, $1 \leqq \nu \leqq p + 1$, provided n is large.

Fix $\varepsilon > 0$, and ν, $1 \leqq \nu \leqq p + 1$. If $z \in \partial X$, then for each $n \geqq 1$,

$$\frac{|z - y_n|}{|x - y_n|} \geqq K,$$

by (3). Choose $r_1 > 0$ such that

$$\mu B(x, r_1) < \frac{\varepsilon K}{2}.$$

Choose $r > 0$ such that

$$\frac{r}{r_1} < \min\left\{\frac{\varepsilon}{2^{\nu+1} \| \mu \|}, \frac{1}{2}\right\}.$$

Then choose N so large that $n \geqq N$ ensures $|x - y_n| < r$. Then, for $n \geqq N$,

$$\begin{aligned} |x - y_n|^\nu \mu^\nu(y_n) &= |x - y_n|^\nu \int \frac{d\,|\mu|(z)}{|z - y_n|^\nu} \\ &= |x - y_n|^\nu \left\{\int_{C \setminus B(x, r_1)} + \int_{B(x, r_1)}\right\} \\ &\leqq \frac{r^\nu \| \mu \|}{(r_1/2)^\nu} + \frac{\mu B(x, r_1)}{K^\nu} \\ &\leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This completes the proof.

COROLLARY 1. *Suppose* $\mathring{X}$ *satisfies a cone condition at a point* $x \in \partial X$. *Then whenever* $p = 0$ *and* x *is not a peak point for* $R(X)$, *or* $p \geqq 1$ *and* $R(X)$ *admits a bounded* pth *order point derivation at* x, *it follows that* $d^p(y, x) \to 0$ *as* y *approaches* x *along the midline of the cone.*

This clearly follows from Theorem 3. Using the language of tangent cones [3, p. 233] we can say more.

COROLLARY 2. *Let* $x \in \partial X$, E *be a compact connected subset of* X, $x \in E$, $E\backslash\{x\} \subset \mathring{X}$, *and suppose that*

$$\operatorname{Tan}(E, x) \cap \operatorname{Tan}(\partial X, x) = (0) .$$

Then under the same hypothesis on p, $R(X)$ *as before,* $d^p(y, x) \to 0$ *as* y *approaches* x *in* E.

COROLLARY 3. *Suppose* $\mathring{X}$ *satisfies a cone condition at* x, *and* Γ *is the midline of the cone. Suppose* $R(X)$ *admits a bounded* p^{th} *order point derivations at* $x (p \geqq 1)$. *Let* D_x^p *and* D_x^{p-1} *denote the normalized point derivations of orders* p *and* $p - 1$ *at* x. *Then*

$$D_x^p f = \lim_{\substack{y \to x \\ y \in \Gamma}} \left[\frac{f^{(p-1)}(y) - D_x^{p-1} f}{y - x} \right]$$

for every $f \in R(X)$, *and the convergence is equiconvergence on* $R(X, 1)$.

This follows readily from Corollary 1.

6. For examples to which these results apply, see [5], [10]. Hallstrom [6] has given necessary and sufficient conditions that $R(X)$ admit a bounded point derivation at a point x. Essentially, the complement of X has to be "thin" at x, in terms of analytic capacity.

Let a_n, r_n be two sequences of positive numbers such that

$$1 > a_n + r_n > a_n > a_n - r_n > a_{n+1} + r_{n+1} ,$$

for $n = 1, 2, 3, \cdots$. Let D_n denote the open disc with centre a_n and radius r_n. Let X be the compact set obtained by removing $\bigcup_{n=1}^{+\infty} D_n$ from the closed unit disc D. X is an example of a so-called L-set.

For these L-sets, the point 0 is a peak point for $R(X)$ if and only if $\sum_{n=1}^{+\infty} r_n/a_n = +\infty$ [10], and $R(X)$ admits a p^{th} order bounded point derivation at 0 provided $\sum_{n=1}^{+\infty} r_n/(a_n^{p+1}) < +\infty$. Let E denote the negative real axis. Applying Corollary 1 to X we obtain the following:

THEOREM 4.

(1) *Suppose* $\sum_{n=1}^{+\infty} r_n/a_n < +\infty$. *Then* $\lim_{\substack{z\to 0\\ z\in E}} (d^0(z, 0) = 0$.

(2) *Suppose* $\sum_{n=1}^{+\infty} r_n/(a_n^{p+1}) < +\infty$. *Then* $\lim_{\substack{z\to 0\\ y\in E}} d^p(z, 0) = 0$.

By choosing, say, $a_n = 1/(n+1)$, $r_n = 1(n+1)!!$ we can ensure that the hypothesis of (2) is satisfied for every $p \leqq 0$, so that $f^{(p)}z$ is equiconvergent to $f^{(p)}(0)$ on $R_0(X, 1)$, for every p.

One might wonder whether some kind of Browder density theorem might work for $p > 0$: if $R(X)$ admits a pth order bounded point derivation at x, are there always other bounded derivations at nearby points? The answer is no: in [9] an example is constructed in which $R(X)$ admits a first order bounded point derivation at just one point. Moreover, this example can be modified to produce an example with a bounded point derivation of every order at that certain point, and no other bounded point derivations of any order $\geqq 1$ anywhere else.

What goes wrong? The following observation may clarify things. If μ represents a first order bounded point derivation on $R(X)$ at x and $\mu^2(y) < +\infty$, set

$$C = \int \frac{(z-x)^2}{z-y}\, d\mu(z) ,$$

$$D = \int \frac{(z-x)^2}{(z-y)^2}\, d\mu(z) .$$

Then, provided $C \neq 0$ and $D \neq 0$, the measure

$$V = \left\{\frac{1}{C}\frac{(z-x)^2}{(z-y)^2} - \frac{1}{CD}\frac{(z-x)^2}{z-y}\right\}\mu$$

represents a first order bounded point derivation on $R(X)$ at y. So this gives a sufficient condition for the existence of other derivations: $\{y \mid \mu^2(y) < +\infty, C \neq 0, D \neq 0\} \neq \varnothing$. Unfortunately μ^2 is the potential associated with harmonic functions in R^4, and the associated capacity, C^2, vanishes on planar sets. So it is entirely possible, even likely, that $\mu^2(y) \equiv +\infty$ on spt μ. In fact, $\mu^2(y) < +\infty$ if and only if

$$\sum_{n=1}^{+\infty} 4^n \,|\mu|\, (A_n(y)) < +\infty ,$$

where $A_n(y) = \{z \mid 1/2^{n+1} \leqq |z-y| \leqq 1/2^n\}$, $n = 1, 2, 3, \cdots$. Thus, for instance, if

$$\overline{\lim_{r\to 0}}\, \frac{|\mu|\,(B(y, r))}{r^2} > 0$$

(i.e., $|\mu|$ has positive area density at y), then $\mu^2(y) = +\infty$.

Returning to the problem posed in § 1, we note that for $x \in \partial(\mathring{X})$, without some condition on $\mathring{X}$, we cannot ensure that there will be a sequence $x_n \to x$ with $x_n \in \mathring{X}$ and $f'(x_n)$ equiconvergent to $f'(x)$ on $R_0(X, 1)$, even when $s^1(x) < +\infty$. For let X be the example of [9], with a bounded point derivation just at 0, and select any sequence $\{x_n\}$ of distinct points of X, tending to 0. For each $n (n = 1, 2, 3, \cdots)$ there is a function $f_n \in R_0(X, 1)$ such that $f_n'(x_n) > 4n$. Inductively, choose a closed disc D_n centered at x_n such that f_n is analytic in a neighborhood of D_n, $\|f_n\|_{D_n} \leqq 2$, $|f_n'(z)| > 2n$ for $z \in D_n$, $D_n \cap D_m = \varnothing$ for $m < n$, $x_m \notin D_n$ for $m > n$. Form a new compact set $Y = X \cup (\bigcup_{n=1}^{+\infty} D_n)$. Then $R(Y)$ still admits a bounded point derivation at 0. The only other bounded point derivations are at points of the D_n. For $z \in D_n$, $s^1(z) > n$. So there is no sequence of points of $\mathring{Y} = \bigcup_{n=1}^{+\infty} \mathring{D}_n$ along which f' is equiconvergent to $f'(0)$ on $R_0(Y, 1)$.

7. Let X be a compact subset of the plane. Let A be an algebra of functions on C which contains the polynomial and all of whose functions are analytic on $\mathring{X}$. Suppose A, regarded as a subset of $C(X)$, forms a function algebra. Suppose A enjoys the *Arens property*: For each $x \in X$,

$$A_x = \{f \in A \mid f \text{ is analytic on a neighborhood of } x\}$$

is dense in A in the uniform norm on X. (A sufficient condition for this is that A contains a dense subset B which is "*T_ϕ-invariant*", i.e., the function $T_\phi f$, given by

$$T_\phi f(z) = \frac{1}{\pi} \int \frac{f(z) - f(\zeta)}{z - \zeta} \frac{\partial \phi(\zeta)}{\partial \bar{\zeta}} \, d\mathscr{L}^2(\zeta)$$

belongs to B whenever f belongs to B and ϕ is a continuously differentiable function with compact support. An example is $A = A(X)$, the algebra of all continuous functions on C which are analytic on $\mathring{X}$; another example is $A = A^\alpha(X)$, the uniform closure on X of those functions in $A(X)$ which satisfy a condition Lip α on C.) Then most of what we have done for $R(X)$ goes through for A. New functions d^p, s^p, d_0^p may be defined analogously, for instance:

$$d^p(x, y) = \sup \{|f^{(p)}(x) - f^{(p)}(y)| \mid f \in A, \|f\|_X \leqq 1, \\ f \text{ is analytic on a neighborhood of } \{x, y\}\} .$$

For any $x \in C$ we can form A_x. So given any compact set $Y \subset C$ we may form a new algebra

$$Y(A) = \bigcap_{x \in Y} (\text{Uniform closure on } Y \text{ of } A_x \cap A(Y)) .$$

$Y(A)$ is clearly a uniform algebra on Y, contains the polynomials, and all its functions are analytic on $\mathring{Y}$. Moreover, by its definition, it has the Arens property.

Replacing $R(X)$ by A, Theorem 1 will go through, except that (7) will have to be changed:

(7′) *if* $x \in \partial X$, V_n *is a decreasing sequence of compact neighborhoods of* x, *whose intersection is* $\{x\}$, *and* $X_n = X \cup V_n$, *then* $s_n^p(x) \uparrow s^p(x)$, *and* $d_n^p(x, \cdot) \uparrow d^p(x, \cdot)$, *where* s_n^p *and* d_n^p *are the* s^p *and* d^p *functions associated with the algebras* $X_n(A)$.

Lemma 1 goes through, using the Arens property.

The maximal ideal space of A is X (cf. [1], its Šilov boundary is a subset of ∂X, so Theorems 2 and 3 work for A in place of $R(X)$.

8. Now we turn to $H^\infty(U)$, the Banach algebra of *bounded analytic functions* (with L^∞ norm) on the bounded open set $U \subset \boldsymbol{C}$. First, we look at $H^\infty(U)$ itself. There is a natural projection map from the maximal ideal space $\mathscr{M}$ of $H^\infty(U)$ to $\bar{U}$, given by $\phi \rightarrow \phi(z)$ (recall that z denotes the identity map of $\boldsymbol{C}$). The fiber $\mathscr{M}_x$ over a point $x \in U$ consists of one point ϕ_x = evaluation at x. The fiber $\mathscr{M}_x$ over a point $x \in \partial U$ is usually very large. Gamelin and Garnett [5] showed that a necessary and sufficient condition for $\mathscr{M}_x$ to be a *peak set* for $H^\infty(U)$ is that

$$\sum_{n=1}^{+\infty} 2^n \gamma(A_n(x) \backslash U) = +\infty \,. \tag{4}$$

Here γ denotes the analytic capacity:

$$\gamma(K) = \sup \{ | f'(\infty) | \mid f \text{ is analytic off } K, \| f \| \leqq 1, f(\infty) = 0 \} \,.$$

When $\mathscr{M}_x$ is not a peak set, they showed that it contains a *distinguished homomorphism*, ϕ_x, characterized by the property that it has a representing measure on $\mathscr{M}$ with no mass on $\mathscr{M}_x$.

We say that an element $D \in H^\infty(U)^*$, a continuous linear map of $H^\infty(U)$ to $\boldsymbol{C}$, is a *first order bounded point derivation* at a point $\phi \in \mathscr{M}$ if

$$D(fg) = \phi(f)Dg + \phi(g)Df$$

whenever $f, g \in H^\infty(U)$. D is called *regular* if $Dz \neq 0$, and a regular D is *normalised* if $Dz = 1$. We shall be concerned with regular derivations only, but we note that there are usually many derivations on $H^\infty(U)$ which annihilate z. For instance, let U be the open unit disc. Then Hoffman [7] has shown that the fiber $\mathscr{M}_1$ over the point $1 \in \partial U$ contains many homeomorphic images of the unit disc, on each of which all the functions in $H^\infty(U)$ are analytic. So there is a superabundance of bounded point derivations at points of $\mathscr{M}_1$, and each of these derivations annihilates z.

Inductively, we say $H^\infty(U)$ admits a regular normalized pth order bounded point derivation at $\phi \in \mathscr{M}$ if the following hold:

(1) For each ν, $1 \leqq \nu \leqq p-1$, D^ν is a νth order regular normalized bounded point derivation at ϕ.

(2) There is an element $D^p \in H^\infty(U)^*$ such that

$$D^p(fg) = \sum_{\nu=0}^{p} \binom{p}{\nu} D^\nu f D^{p-\nu} g ,$$

for all $f, g \in H^\infty(U)$, where $D^0 f$ means $\phi(f)$.

(3) $D^p z^p = p!$

We observe that for $p \geqq 1$ there cannot be any regular pth order bounded point derivation at a point $\phi \in \mathscr{M}_x \backslash \{\phi_x\}$. For such a derivation would have a representing measure μ on $\mathscr{M}$, and then $((z-x)^p/p!)\mu$ would be a representing measure for ϕ with no mass on $\mathscr{M}_x$, which is impossible.

THEOREM 5. *Let $x \in U$, $p \geqq 1$. Then $H^\infty(U)$ admits a regular bounded* p*th order point derivation at the distinguished homomorphism ϕ_x in the fiber over x if and only if*

$$\sum_{n=1}^{+\infty} 2^{(p+1)n} \gamma(A_n(x) \backslash U) < +\infty . \tag{5}$$

Proof. If (5) holds, then certainly (4) fails, so $\mathscr{M}_x$ is not a peak fiber and ϕ_x exists. By a device in Gamelin and Garnett's proof of the peak set criterion [5, p. 459, third paragraph], U can be shrunk a little to produce a compact set X with the properties:

(1) $X \subset U \cup \{x\}$,

(2) $x \in X$,

(3) $\sum_{n=1}^{+\infty} 2^{(p+1)n} r(A_n(x) \backslash X) < +\infty$.

By Hallstrom's Theorem [6, p. 156], $R(X)$ admits a (normalized) bounded point derivation of order p at x. Choose a representing measure μ for this derivation with support on X *and no mass at* x. Then, for $\nu = 0, 1, \cdots, p$ the measure $\mu_\nu = (\nu!(z-x)^{p-\nu}/p!)\mu$ represents a (normalized) νth order bounded point derivation on $R(X)$ at x, if $\nu \geqq 1$, and μ_0 represents x and *has no mass at* x. Now any function in $H^\infty(U)$ which extends analytically to a neighborhood of x belongs to $R(X)$, so for any two such functions, f and g, we have

$$\int fg\,d\mu = \sum_{\nu=0}^{p} \binom{p}{\nu} \int f\,d\mu_\nu \int g\,d\mu_{p-\nu} . \tag{6}$$

Since, as is well-known [5, Cor. 2.2], the set of all such functions is pointwise boundedly dense in $H^\infty(U)$, the dominated convergence theorem implies that (6) holds for any $f, g \in H^\infty(U)$. Thus μ represents a regular bounded pth order point derivation on $H^\infty(U)$ at ϕ_x.

For the other direction, assume (5) fails. If $\mathscr{M}_x$ is a peak set there is no distinguished homomorphism, and nothing to prove. Otherwise, (4) fails, and we may, just as in Hallstrom's proof of his Theorem 1′ [6, pp. 163–164], construct a sequence of functions g_n, each one in $H^\infty(U)$ and analytic in a neighborhood of x such that $|g_n^{(p)}(x)| > n\|g\|_\infty$. Thus $H^\infty(U)$ cannot admit a p^{th} order bounded point derivation at ϕ_x. This proves the theorem.

We remark that there is at most one regular normalised bounded pth order point derivation at a distinguished homomorphism ϕ_x. For, from the proof of Theorem 5, any two agree on a dense subset of $H^\infty(U)$, and have representing measures with no mass on $\mathscr{M}_x$. Thus, by dominated convergence, they coincide.

9. The zero order Gleason metric d^0 on the maximal ideal space of $H^\infty(U)$ is given by

$$d^0(\phi, \psi) = \sup\{|\phi(f) - \psi(f)| \mid f \in H^\infty(U), \|f\|_U \leqq 1\}.$$

To define the higher order metrics, we take first the case where ϕ and ψ are distinguished homomorphisms at each of which $H^\infty(U)$ admits normalised regular bounded p^{th} order point derivations D_ϕ^p and D_ψ^p. Then

$$d^p(\phi, \psi) = \sup\{|D_\phi^p f - D_\psi^p f| \mid f \in H^\infty(U), \|f\|_U \leqq 1\}.$$

In all other cases, we set $d^p(\phi, \psi) = +\infty$. Let $s^p(\phi)$ be the norm of D_ϕ^p, if this exists, otherwise $s^p(\phi) = +\infty$. For points $y \in U$ we will write y for "evaluation at y".

THEOREM 6. *Let $p \geqq 1$. Suppose there is a constant $K > 0$ and a sequence of points $y_n \in U$, $|y_n - x| \to 0$ as $n \to +\infty$, such that*

$$\operatorname{dist}[y_n, \partial U] \geqq K|y_n - x|.$$

Suppose $H^\infty(U)$ admits a regular p^{th} order bounded point derivation at the distinguished homomorphism ϕ_x over x. Then $d^p(y_n, \phi_x) \to 0$ as $n \to +\infty$.

Proof. We shall deduce this from Theorem 3. As in Theorem 5, we may shrink U to a compact set X which satisfies the hypotheses of Theorem 3, with a smaller K. Thus there are representing measures μ_n for the $D^p_{y_n}$, and μ for $D^p_{\phi_x}$, with closed support in $U \cup \mathscr{M}_x$ and no mass on $\mathscr{M}_x$ such that

$$\int f\,d\mu_n \longrightarrow \int f\,d\mu$$

uniformly for $f \in R(X)$. Again, since $R_0(X)$ is pointwise boundedly dense in $H^\infty(U)$, this means that $D^p_{y_n} f = \int f\,d\mu_n$ is equiconvergent to $D^p_{\phi_x} f = \int f\,d\mu$ for all $f \in H^\infty(U)$.

The analogous result when $p = 0$ (also a corollary of Theorem 3) is due to Gamelin and Garnett [5, 5.1].

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