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14.—A Generalised Walsh-Lebesgue Theorem.* By A. G. O'Farrell,

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Synopsis

Let X be the boundary of a compact set which does not separate the plane, $\boldsymbol{C}$. Let Φ and Ψ be homeomorphisms of $\boldsymbol{C}$ to $\boldsymbol{C}$ with opposite orientations. Then every continuous complex-valued function on X is the uniform limit on X of sums $p(\Phi)+q(\Psi)$, where p and q are analytic polynomials.

1. For a compact subset X of the complex plane $\boldsymbol{C}$, $C(X)$ denotes the uniform algebra of all continuous complex-valued functions on X. If $A \subset C(X)$, then $[A]_X$ denotes the closure of A in the uniform norm of $C(X)$. If $f_1, f_2, \ldots, f_n \in C(X)$, then $P(f_1, f_2, \ldots, f_n)$ denotes the algebra of all polynomials in $f_1, f_2, \ldots, f_n$ with complex coefficients. The following theorem of Browder and Wermer appears in [**1**].

Theorem. *Let Z and Ψ denote, respectively, the identity map of the unit circle $S \subset \boldsymbol{C}$, and a direction-reversing homeomorphism of S to itself. Then*

$$[P(Z)+P(\Psi)]_S = C(S).$$

This result fits into a circle of ideas concerned with removing the non-topological assumption of closure under complex conjugation from the complex Stone-Weierstrass Theorem and related results [**3**]. In this paper we exploit the Browder-Wermer Theorem and some abstract techniques to obtain results in which topological conditions imply approximation theorems.

Our main theorem is the following.

Theorem. *Let X be the boundary, ∂Y, of a compact set Y with connected complement in $\boldsymbol{C}$. Let Φ and Ψ be homeomorphisms of $\boldsymbol{C}$ to $\boldsymbol{C}$ with degree $\Phi = -$degree $\Psi = 1$. Then*

$$[P(\Phi)+P(\Psi)]_X = C(X).$$

Proof. We may assume that $\Phi = Z$, and we do.

Let μ be a measure supported on X which annihilates $P(Z)+P(\Psi)$. Then $\Psi_\# \mu$ annihilates $P(Z)$ and is supported on $\Psi(X)$ (recall that $\Psi_\# \mu$ is the measure defined by $\int f d\Psi_\# \mu = \int f \circ \Psi d\mu$). Let $\{U_i\}$ be the family of components of the interior of Y, and for each i, fix a point $x_i \in U_i$. By the decomposition lemma for orthogonal measures [4, §5],

$$\mu = \Sigma \mu_i,$$

$$\Psi_\# \mu = \Sigma \lambda_i,$$

where for each i, μ_i is supported on ∂U_i, λ_i is supported on $\partial \Psi(U_i)$, and both μ_i and λ_i annihilate $P(Z)$. Also μ_i is absolutely continuous with respect to the harmonic

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measure for x_i on ∂U_i, and λ_i is absolutely continuous with respect to the harmonic measure for $\Psi(x_i)$ on $\partial\Psi(U_i)$. Let $\nu_i = \Psi_{\#}^{-1}\lambda_i$, so that

$$\mu = \Sigma \nu_i.$$

By Fatou's theorem, the harmonic measure for U_i is supported on the set of points of ∂U_i which are accessible from U_i, hence μ_i and ν_i are supported on this set. Also, μ_i and ν_i contain no point masses, since no harmonic measure contains point masses. If $i \neq j$, then $\partial U_i \cap \partial U_j$ contains at most one point which is accessible from both U_i and U_j, since Y has connected complement. Hence

$$\mu_i(\partial U_i \cap \partial U_j) = \nu_i(\partial U_i \cap \partial U_j) = 0.$$

Let χ_i denote the characteristic function of ∂U_i. Then $\chi_i\mu_j = \chi_i\nu_j = 0$ whenever $i \neq j$, hence for each i,

$$\mu_i = \chi_i\mu_i = \sum_j \chi_i\mu_j = \chi_i\mu$$
$$= \sum_j \chi_i\nu_j = \chi_i\nu_i = \nu_i.$$

Now fix i, and let ϕ and ψ be conformal maps of the unit disc D to U_i and $\Psi(U_i)$, respectively, such that $\phi(0) = x_i$ and $\psi(0) = \Psi(x_i)$. By the decomposition lemma $\mu_i = h\sigma$, where σ is the harmonic measure on ∂U_i for x_i and h belongs to the abstract Hardy space $H_0^1(\sigma)$, that is h belongs to the closure of $P(Z)$ in $L^1(\sigma)$ and $\int h d\sigma = 0$. The map ϕ has an extension as a measurable one-to-one map of a set of full $d\theta$ measure on S on to a set of full harmonic measure on ∂U_i, with $\phi_{\#}(d\theta/2\pi) = \sigma$. Choose polynomials p_n so that $p_n \to h$ in $L^1(\sigma)$ and $\int p_n d\sigma = 0$. Then $p_n \circ \phi \to h \circ \phi$ in $L^1(d\theta)$ and $p_n \circ \phi(0) = 0$. Hence $\alpha = \phi_{\#}^{-1}\mu_i$ annihilates $P(Z)$. Similarly, $\beta = \psi_{\#}^{-1}\Psi_{\#}\mu_i$ annihilates $P(Z)$. Consider the map $\Psi_1 = \psi^{-1} \circ \Psi \circ \phi$, which maps D on to itself. Since Ψ carries prime ends (cf. [2]) on ∂U_i onto prime ends on $\partial\Psi(U_i)$, it is clear that Ψ_1 extends to a $1-1$, onto, direction-reversing, and hence bicontinuous, map of S to itself. Furthermore,

$$\beta = \psi_{\#}^{-1}\Psi_{\#}\mu_i = \psi_{\#}^{-1}\Psi_{\#}\phi_{\#}\alpha = \Psi_{1\#}\alpha,$$

so that α annihilates $P(Z)+P(\Psi_1)$. By the Browder-Wermer Theorem, $\alpha = 0$, hence $\mu_i = 0$.

Since this is true for each i, we conclude that $\mu = 0$, so that $P(Z)+P(\Psi)$ is dense.

Comment. The classical Walsh-Lebesgue Theorem [4], states that if X is the boundary of a compact set with connected complement, then the harmonic polynomials are dense in all the continuous functions on X. This assertion is just

$$[P(Z)+P(\bar{Z})]_X = C(X)$$

in our notation, so that our theorem is a generalisation.

Before turning to the applications we pause and note some equivalents of the hypothesis.

PROPOSITION. *Let X be a compact subset of $\mathbf{C}$ and let $\mathcal{U}$ be the family of bounded components of $\mathbf{C} \sim X$. The following three conditions are equivalent.*

(1) *$X \cap U = \emptyset$ whenever T is a subset of X which separates $\mathbf{C}$ and U is a bounded component of $\mathbf{C} \sim T$;*

(2) $X = \partial(X \cup (\cup \mathscr{U}))$;

(3) $X = \partial Y$ *for some compact set* Y *with connected complement.*

Proof. (1)⇒(2). Suppose (1) holds, and let $Y = X \cup (\cup \mathscr{U})$. Then $\mathbf{C} \sim Y$ is the unbounded component of $\mathbf{C} \sim X$, hence $\partial Y \subset X$. Fix $x \in X$. Then $x \in \text{clos}(\mathbf{C} \sim Y)$ for otherwise x lies in a bounded component of $\partial(\mathbf{C} \sim Y) \subset X$ and (1) is contradicted. Hence $x \in \text{clos}(\mathbf{C} \sim Y) \cap X \subset \text{clos}(\mathbf{C} \sim Y) \cap \text{clos}\, Y = \partial Y$. Thus $X \subset \partial Y$ and (2) holds.

(2)⇒(3) is trivial.

(3)⇒(1). If $X = \partial Y$, where $\mathbf{C} \sim Y$ is connected, and if T and U are as in (1), then $\mathbf{C} \sim Y$ is contained in the unbounded component of $\mathbf{C} \sim T$, so $\mathbf{C} \sim Y$ does not meet U. Thus $U \subset Y$, so $U \cap X$ is empty.

For brevity, let $\mathscr{K}$ denote the class of compact sets X for which the equivalent conditions (1)–(3) of the proposition hold.

2. We draw a corollary from the first section of the proof of the theorem. It enables us to patch together sets for which we know

$$[P(\Phi)+P(\Psi)]_X = C(X)$$

to get others.

COROLLARY 1. *Let* $Y \subset \mathbf{C}$ *be compact,* $\mathbf{C} \sim Y$ *be connected,* Φ *and* Ψ *be* $1-1$ *maps in* $C(Y)$, *and let* $\{U_i\}$ *be the family of components of the interior of* Y. *Suppose that*

$$[P(\Phi)+P(\Psi)]_{\partial U_i} = C(\partial U_i)$$

for each i. *Then*

$$[P(\Phi)+P(\Psi)]_{\partial Y} = C(\partial Y).$$

The following classical theorem provides additional input for Corollary 1. It was brought to our attention by B. Cole.

THEOREM. *Let* Φ *be a singular homeomorphism of* S *to* S. *Then*

$$[P(Z)+P(\Phi)]_S = C(S).$$

Proof. Let μ annihilate $P(Z)+P(\Phi)$. Then, by the F. and M. Riesz Theorem, μ and $\Phi_*\mu$ are absolutely continuous with respect to linear measure on S. Hence $\mu = 0$.

More generally, we say a homeomorphism Φ is singular on the boundary of the connected open set U if Φ carries a set of full harmonic measure on ∂U to a set of zero harmonic measure on $\partial\Phi(U)$. By imitating the technique of the main theorem one can show the following.

COROLLARY 2. *Let* $X \in \mathscr{K}$, *and let* Φ *be a homeomorphism of* $\mathbf{C}$ *to* $\mathbf{C}$ *which is singular on the boundary of each bounded component of* $\mathbf{C} \sim X$. *Then*

$$[P(Z)+P(\Phi)]_X = C(X).$$

Clearly, one can ring the changes here, combining singular maps with maps of negative degree in various combinations.

The remaining application is the invention of K. Preskenis. The author wishes to thank him for several conversations in which he set forth his ideas. These results concern sets with possibly non-empty interior.

COROLLARY 3. *Suppose F is a real-valued continuous function on the compact set $X \subset \mathbf{C}$ such that each level set of F is an element of $\mathscr{K}$. Let Φ and Ψ be homeomorphisms of $\mathbf{C}$ to $\mathbf{C}$ such that degree $\Phi = -$degree Ψ. Then*

$$[P(\Phi, F)+P(\Psi, F)]_X = C(X).$$

Remark. One may replace the assumption that Φ and Ψ have opposite degree by a singularity assumption on $\Psi \circ \Phi$; or by a mixture of the two.

This result may be deduced from our theorem by direct methods. A more elegant approach uses the following theorem of de Branges (see [**3**, Lemma 2.3]; this is a slight generalisation of that lemma, but the same proof works).

THEOREM. *Let X be a compact Hausdorff space, let A be an algebra of continuous functions on X which contains the constants, and let B be a subspace of $C(X)$ which is an A-module. Suppose A contains a real-valued function F. Then the extreme annihilating measures of B in the unit ball of $C(X)^*$ are supported on the level sets of F.*

Proof of Corollary 3. Let μ be an extreme annihilating measure of $P(\Phi, F)+P(\Psi, F)$ in the ball of $C(X)^*$. Then by de Branges' Theorem μ sits on some contour of F, and by the main theorem, $\mu = 0$. Hence the Krein-Milman Theorem implies there are no annihilators except the zero measure.

We obtain a corollary resembling a theorem of Mergelyan [**3**, Theorem 2.1].

COROLLARY 4. *If Ψ is a homeomorphism of $\mathbf{C}$ to $\mathbf{C}$ of degree -1 and $[P(Z, \Psi)]_X$ contains a real-valued function F whose level sets are elements of $\mathscr{K}$, then*

$$[P(Z, \Psi)]_X = C(X). \tag{*}$$

It is an open question whether the assumption that Ψ is an orientation-reversing homeomorphism of $\mathbf{C}$ to $\mathbf{C}$ is sufficient by itself to guarantee the equality (*) [cf. **3**].

COROLLARY 5. *Let Φ, Ψ and Λ be homeomorphisms of $\mathbf{C}$ to $\mathbf{C}$ with degree $\Phi = -$degree Ψ, and let α be a positive real number. Then for any compact set $X \subset \mathbf{C}$ we have*

$$[P(\Phi, |\Lambda|^\alpha)+P(\Psi, |\Lambda|^\alpha)]_X = C(X).$$

Proof. The contours of $|\Lambda|^\alpha$ are closed Jordan curves, so they are elements of $\mathscr{K}$, and Corollary 3 applies.

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