TWENTY FIFTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 12 May 2012

Second Paper

Time allowed: **Three hours.**

6. Let $S(n)$ be the sum of the decimal digits of n. For example, $S(2012) = 2+0+1+2 = 5$. Prove that there is no integer $n > 0$ for which $n - S(n) = 9990$.

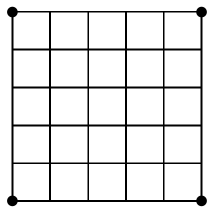
7. Consider a triangle ABC with $|AB| \neq |AC|$. The angle bisector of the angle CAB intersects the circumcircle of $\triangle ABC$ at two points A and D. The circle of centre D and radius $|DC|$ intersects the line AC at two points C and B'. The line BB' intersects the circumcircle of $\triangle ABC$ at B and E.

   Prove that B' is the orthocentre of $\triangle AED$.

8. Suppose a, b, c are positive numbers. Prove that
$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 1\right)^2 \geq (2a + b + c)\left(\frac{2}{a} + \frac{1}{b} + \frac{1}{c}\right),$$
with equality if and only if $a = b = c$.

9. Let $x > 1$ be an integer. Prove that $x^5 + x + 1$ is divisible by at least two *distinct* prime numbers.

10. Let n be a positive integer. A mouse sits at each corner point of an $n \times n$ square board, which is divided into unit squares as shown below for the example $n = 5$.



The mice then move according to a sequence of *steps*, in the following manner:

(a) In each step, each of the four mice travels a distance of one unit in a horizontal or vertical direction. Each unit distance is called an *edge* of the board, and we say that each mouse *uses* an edge of the board.
(b) An edge of the board may not be used twice in the same direction.
(c) At most two mice may occupy the same point on the board at any time.

The mice wish to collectively organise their movements so that each edge of the board will be used twice (not necessarily by the same mouse), and each mouse will finish up at its starting point. Determine, with proof, the values of n for which the mice may achieve this goal.