TWENTY FIFTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 12 May 2012

Second Paper

Solutions

6. Proposed by Gordon Lessells.

Let $S(n)$ be the sum of the decimal digits of n. For example, $S(2012) = 2+0+1+2 = 5$. Prove that there is no integer $n > 0$ for which $n - S(n) = 9990$.

Solution

Assume there exists n for which $n - S(n) = 9990$. As $n - S(n) \leq n$, we may assume $n \geq 9\,990$. For integers $n \leq 99\,999$, $S(n) \leq 45$. Thus if $n - S(n) = 9\,990$, $n \leq 10\,035$ or $n \geq 100\,000$. For $n \geq 100\,000$, $n - S(n) \geq 99,000$. For $10\,000 \leq n \leq 10\,035$, $n - S(n) \geq 9\,999$. For $9\,990 \leq n \leq 9\,999$, $n - S(n) = 9\,963$. Hence $n - S(n)$ cannot take the value $9\,990$.

Alternatively, one may write $n = \sum_{i=0}^{k} a_i 10^i$ with $a_i \leq 9$, $a_k > 0$ and observe that $n - S(n) = \sum_{i=1}^{k} a_i N_i \geq N_k$, where $N_i = 999\ldots99$ consists of i digits, all equal to 9. If $k \geq 4$, we have $n - S(n) \geq N_4 = 9\,999$. If $k \leq 3$, we have $n - S(n) \leq 9N_3 + 9N_2 + 9N_1 = 81 \cdot (111 + 11 + 1) = 81 \cdot 123 = 9\,963$. This shows that there does not exist an integer n with $n - S(n) = 9\,990$.

7. Proposed by Anca Mustata.

Consider a triangle ABC with $|AB| \neq |AC|$. The angle bisector of the angle CAB intersects the circumcircle of $\triangle ABC$ at two points A and D. The circle of centre D and radius $|DC|$ intersects the line AC at two points C and B'. The line BB' intersects the circumcircle of $\triangle ABC$ at B and E.

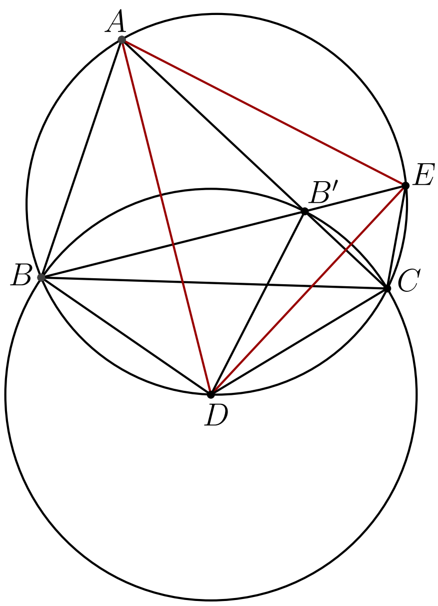
Prove that B' is the orthocentre of $\triangle AED$.

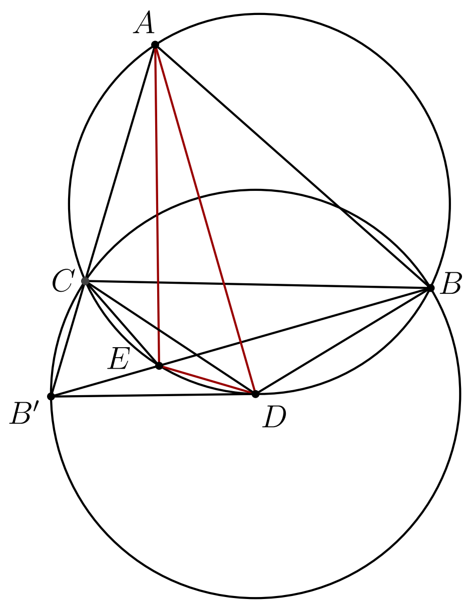
Solution 1.

We will prove that $\triangle ABD \equiv \triangle AB'D$, hence $|AB| = |AB'|$ and $|BD| = |B'D|$ and so AD is the perpendicular bisector of segment BB'. Similarly, we will prove $|EC| = |EB'|$ and as we know $|DC| = |DB'|$, it will follow that ED is the perpendicular bisector of segment $B'C$. Hence, the lines $EB \perp AD$ and $AC \perp ED$ are altitudes in $\triangle AED$, intersecting at B'.

Indeed, $\triangle ABD \equiv \triangle AB'D$ will follow by $A.S.A$ from the equality $\widehat{ABD} = \widehat{AB'D}$, as we already know that the two triangles have a common side AD and pairs of equal angles $\widehat{BAD} = \widehat{B'AD}$. As well, $|EC| = |EB'|$ will follow from the angle equality $\widehat{B'CE} = \widehat{EB'C}$.

It remains to prove $\widehat{ABD} = \widehat{AB'D}$ and $\widehat{B'CE} = \widehat{EB'C}$. These proofs are slightly diagram dependent, so we will distinguish two cases:





Case I: $|AB| < |AC|$ so that $\widehat{ABD} > \widehat{ACD}$ and as $\widehat{ABD} + \widehat{ACD} = 180°$, then $\widehat{ACD} < 90°$ and B' is inside the segment AC. The quadrilateral $ABDC$ is cyclic and so

$$\widehat{ABD} = 180° - \widehat{ACD} = 180° - \widehat{CB'D} = \widehat{AB'D}$$

(we used the fact that $\triangle DB'C$ is isosceles).

Also, the quadrilateral $ABCE$ is cyclic and so $\widehat{ABB'} = \widehat{B'CE}$ and on the other hand, since $|AB| = |AB'|$ from $\triangle ABD \equiv \triangle AB'D$ above, $\widehat{ABB'} = \widehat{AB'B}$. Hence

$$\widehat{B'CE} = \widehat{AB'B} = \widehat{EB'C}.$$

<u>Case II:</u> If $|AB| > |AC|$ so that $\widehat{ABD} < \widehat{ACD}$ and so $\widehat{ACD} > 90°$ and B' is outside the segment AC. The quadrilateral $ABDC$ is cyclic and so

$$\widehat{ABD} = 180° - \widehat{ACD} = \widehat{B'CD} = \widehat{CB'D}$$

(as $\triangle DB'C$ is isosceles).

Also, the quadrilateral $ABEC$ is cyclic and so

$$\widehat{ABE} = 180° - \widehat{ACE} = \widehat{B'CE}$$

and on the other hand, since $|AB| = |AB'|$ from the above,

$$\widehat{ABE} = \widehat{CB'E} \text{ so } \widehat{B'CE} = \widehat{CB'E}.$$

Solution 2.

By angle chasing we will prove $\widehat{ADE} = \widehat{ABB'} = 90° - \frac{1}{2}\widehat{A}$. As $\widehat{DAC} = \frac{1}{2}\widehat{A}$, it follows that $\widehat{ADE} + \widehat{DAC} = 90°$ so $AC \perp DE$.

On the other hand, $\widehat{DEB} = \widehat{DAB} = \frac{1}{2}\widehat{A}$ so $\widehat{ADE} + \widehat{DEB} = 90°$ so $AD \perp BE$.

One of the two above may be replaced by showing $\widehat{ADB'} + \widehat{EAD} = 90°$ by angle chasing.

Hence B' is the orthocentre of $\triangle ADE$.

The calculation of $\widehat{ADE}$ is diagram dependent:

<u>Case I:</u> $|AB| < |AC|$. Calculate $\widehat{ADE} = \widehat{ABB'} = \widehat{B} - \widehat{B'BC}$ and

$$\widehat{B'BC} = \frac{1}{2}\widehat{B'DC} = 90° - \widehat{ACD} = 90° - \widehat{C} - \widehat{BCD} = 90° - \widehat{C} - \frac{1}{2}\widehat{A},$$

so $\widehat{ADE} = \widehat{ABB'} = \widehat{B} - 90° + \widehat{C} + \frac{1}{2}\widehat{A} = 90° - \frac{1}{2}\widehat{A}$.

<u>Case II:</u> If $|AB| > |AC|$. $\widehat{ADE} = \widehat{ABB'} = \widehat{B} + \widehat{B'BC}$ and

$$\widehat{B'BC} = \frac{1}{2}\widehat{B'DC} = 90° - (180° - \widehat{ACD}) = \widehat{C} + \widehat{BCD} - 90° = \widehat{C} + \frac{1}{2}\widehat{A} - 90°,$$

so $\widehat{ADE} = \widehat{ABB'} = \widehat{B} - 90° + \widehat{C} + \frac{1}{2}\widehat{A} = 90° - \frac{1}{2}\widehat{A}$.

8. Proposed by Finbarr Holland.

Suppose a, b, c are positive numbers. Prove that

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1\right)^2 \geq (2a+b+c)\left(\frac{2}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

with equality if and only if $a=b=c$.

Solution

We derive an equivalent version of the given inequality. To obtain this, expand both sides:

$$\begin{aligned}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1\right)^2 &= 1+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2\\ &= 1+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+2\left(\frac{ab}{bc}+\frac{ca}{ab}+\frac{bc}{ca}\right)+\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\\ &= 1+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+2\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)+\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2},\end{aligned}$$

while

$$\begin{aligned}(2a+b+c)\left(\frac{2}{a}+\frac{1}{b}+\frac{1}{c}\right) &= 4+2\left(\frac{a}{b}+\frac{a}{c}\right)+2\frac{b}{a}+1+\frac{b}{c}+2\frac{c}{a}+\frac{c}{b}+1\\ &= 6+2\left(\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+\frac{c}{a}\right)+\frac{b}{c}+\frac{c}{b}.\end{aligned}$$

Hence the difference between the LHS and the RHS is equal to

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{b}{c}+\frac{c}{b}-5.$$

Now

$$\frac{b}{c}+\frac{c}{b}-2=\frac{b^2+c^2-2bc}{bc}=\frac{(b-c)^2}{bc}\geq 0,$$

and, by the AM-GM inequality for three variables,

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq 3\sqrt[3]{\frac{a^2}{b^2}\frac{b^2}{c^2}\frac{c^2}{a^2}}=3;$$

and these inequalities are strict unless $a=b=c$.

9. Proposed by Tom Laffey.

Let $x > 1$ be an integer. Prove that $x^5 + x + 1$ is divisible by at least two *distinct* prime numbers.

Solution

Step 1. The key idea is to factor the polynomial $f(x) := x^5 + x + 1$. One can do this by observing that if $\omega, \omega^2 = (-1 \pm i\sqrt{3})/2$, are the two primitive cube roots of unity, then $f(\omega) = 0 = f(\omega^2)$, thus giving the factor $(x-\omega)(x-\omega^2) = x^2 + x + 1$ of $f(x)$ and the factorization $f(x) = (x^3 - x^2 + 1)(x^2 + x + 1)$. Alternatively, one can write $f(x) = x^2(x^3 - 1) + (x^2 + x + 1)$, from which the factorization is immediate. More systematically, one can write $f(x) = (x^3 + Ax^2 + Bx + C)(x^2 + Dx + E)$, and determine the integers A, B, C, D, E by comparing the coefficients of corresponding powers of x. The observation that $CE = 1$ implies that $C = E = \pm 1$ simplifies the calculation.

Step 2. Suppose that $x > 1$ is an integer. Then the factors $x^3 - x^2 + 1$ and $x^2 + x + 1$ are both greater then 1, so $f(x)$ is a product of at least two primes.

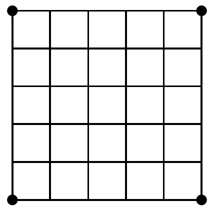
Step 3. It remains to show that at least two distinct primes occur in the product. Suppose p is a prime dividing both $x^2 + x + 1$ and $x^3 - x^2 + 1$. Since

$$x^3 - x^2 + 1 = (x^2 + x + 1)(x - 2) + (x + 3) \quad \text{and}$$
$$x^2 + x + 1 = (x + 3)(x - 2) + 7 \ ,$$

we see that $p = 7$, and also, that at least one of (x^2+x+1) and (x^3-x^2+1) is not divisible by 7^2. So, if $f(x)$ is a power of a prime, then one of x^2+x+1, x^3-x^2+1 is 7 and the other is a power of 7. If $x^2 + x + 1 = 7$, then $x = 2$ and $x^3 - x^2 + 1 = 5$, while $x^3 - x^2 + 1 > 7$ for all $x \geq 3$, so $x^3 - x^2 + 1 = 7$ is impossible. Hence $f(x)$ is not a prime-power and the proof is done.

10. Proposed by Mark Flanagan.

Let n be a positive integer. A mouse sits at each corner point of an $n \times n$ square board, which is divided into unit squares as shown below for the example $n = 5$.



The mice then move according to a sequence of *steps*, in the following manner:

(a) In each step, each of the four mice travels a distance of one unit in a horizontal or vertical direction. Each unit distance is called an *edge* of the board, and we say that each mouse *uses* an edge of the board.

(b) An edge of the board may not be used twice in the same direction.

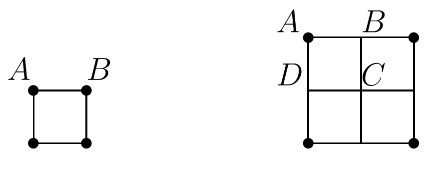
(c) At most two mice may occupy the same point on the board at any time.

The mice wish to collectively organise their movements so that each edge of the board will be used twice (not necessarily by the same mouse), and each mouse will finish up at its starting point. Determine, with proof, the values of n for which the mice may achieve this goal.

Solution

We prove by induction that the mice may achieve this goal for every positive integer n. We will focus on the movements of one mouse starting at one corner A of the board.

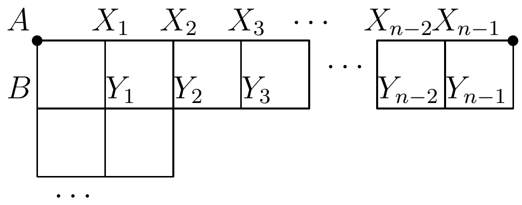
First, for $n = 1$ and $n = 2$, consider the following figures:



For $n = 1$, the mouse moves $A \to B \to A$. Successive mice move in a symmetrical fashion (rotated by 90 degrees).

For $n = 2$, two mice move $A \to B \to C \to B \to A \to D \to A$, and the other two move $A \to D \to A \to B \to C \to B \to A$ (with the understanding that the moves for each successive mouse are rotated by 90 degrees). Note that at four points during this process, two mice occupy the same point of the board at the same time.

Next, for $n > 2$, we assume that a solution exists for an $(n-2) \times (n-2)$ board. Consider the figure below.



The mouse starting at A can execute the moves:

$$
\begin{aligned}
& A \to B \to A \\
& \to X_1 \to Y_1 \to X_1 \\
& \to X_2 \to Y_2 \to X_2 \\
& \to X_3 \to Y_3 \to X_3 \\
& \cdots \to X_{n-1} \to Y_{n-1} \to X_{n-1} \\
& \to X_{n-2} \to X_{n-3} \to X_{n-4} \cdots \to X_1 \to A\,.
\end{aligned}
$$

If the other four mice execute the same moves rotated by 90 degrees, then

(a) Together the mice will use every edge exactly once in each direction, apart from the inner $(n-2) \times (n-2)$ square;

(b) When the mouse starting from A reaches position Y_1, all four mice will be at the corners of the inner $(n-2) \times (n-2)$ square; thus the solution for $n-2$ can be spliced in at this moment before the mice continue on their homeward journey.

It follows by the principle of induction that a solution can be achieved for every positive integer n.